RECEPTION OF DISPERSED
BARKER CODES

by

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ABSTRACT

The Sunblazer radio propagation experiment requires that a signal, which has been transmitted through the solar corona, be detected, and its time of arrival estimated as accurately as possible. A communication scheme which does this optimally is analyzed in terms of its sensitivity to the dispersive effect of the coronal plasma. It is shown that receiver performance can be evaluated if the cross-correlation function of the dispersed and undispersed signal is known. The method of a Taylor expansion of phase is used to obtain an expression for this cross-correlation function that is suitable for machine computation. Three cases are then examined--the rectangular pulse, and the three- and eleven-bit Barker codes. In each case the cross-correlation and the mean-square estimation error are computed and plotted for different amounts of dispersion. The "dispersion time" is defined as a measure of dispersion, and it is seen that signal degradation becomes severe when this parameter becomes larger than the elementary pulse width of the signal.

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I. INTRODUCTION

A primary purpose of the Sunblazer solar probe, which is to be launched in the summer of 1968, is to make measurements of the electron density of the extended solar corona (Refs. 9, 10). This can be done by transmitting two narrowband signals from the satellite on two different frequencies. Each signal will experience a different propagation delay through the corona due to the frequency-dependent group velocities, which in turn are functions of the electron density. The aim, then, is to measure as accurately as possible the relative arrival times of the received signals.

We will see that in designing a system to make this measurement, it is desirable to shape our signal such that it has a peaked auto-correlation function. Barker codes are useful because they have this property. To detect the signal and estimate its arrival time, the optimum receiver takes the correlation function of the demodulated received signal (ideally Barker code plus noise) with another Barker code. The receiver thus produces the auto-correlation function of the Barker code, and signal detection and arrival time estimation is accomplished by measuring the height and time-position of the central peak.

This reception scheme, however, does not take into account the dispersive effect of the coronal plasma on the signal. The demodulated received signal will consist not of a

Barker code plus noise, but a dispersed version of the Barker code plus noise, and the receiver will not calculate the desired auto-correlation function. The performance of the receiver will thus be degraded; finding out to what degree it will be degraded is an objective of this thesis.

An investigation of this dispersion problem had already been begun by Gordon R. Gilbert at the Center for Space Research (Ref. 6). He first applied Ginzburg's phase expansion technique to the correlation reception of a rectangular pulse. This has provided much of the groundwork for my thesis.

A. Sunblazer Radio Propagation Experiment

A wave packet propagating through an ionized medium (e.g. the corona) with no magnetic field present will have a group velocity given by

$$\left(\frac{V_g}{C}\right)^2 = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \tag{1}$$

Here ω is the center frequency of the group, the "plasma frequency" $\omega_p = 2\pi f_p = \sqrt{\frac{ne^2}{m\varepsilon_o}}$, and n is the number of electrons per cc (Ref. 13). We see, then, that the signal velocity, and hence propagation time, is a function of both frequency and electron density. We can use this fact to measure the coronal electron density by transmitting signals on two separate frequencies through the corona and finding the difference in their propagation times. The relative propagation delay between signals on frequencies f_1 and $f_2 > f_1$, will be

$$T = \int \left[\frac{1}{V_{g_1}} - \frac{1}{V_{g_2}}\right] \cdot ds \tag{2}$$

where the line integral is taken along the propagation path. Then,

$$T = \frac{1}{c}\int \left[\frac{1}{\sqrt{1-(f_p/f_1)^2}} - \frac{1}{\sqrt{1-(f_p/f_2)^2}}\right]\cdot ds$$

Or, for f_1 larger than f_p,

$$T \simeq \frac{1}{c}\int \frac{f_p^{\ 2}}{2f_1^{\ 2}} \left[1-\left(\frac{f_1}{f_2}\right)^2\right]\cdot ds \qquad (3)$$

We would like to make T as large as possible so as to maximize the sensitivity of the experiment. Therefore f_1 is made slightly larger than the plasma frequency of the innermost region of the corona, and f_2 is chosen so that $1-(f_1/f_2)^2$ is close to unity. Present plans call for f_1 = 75 Mcps, and f_2 = 225 Mcps (Ref. 9, pps. 38-40).

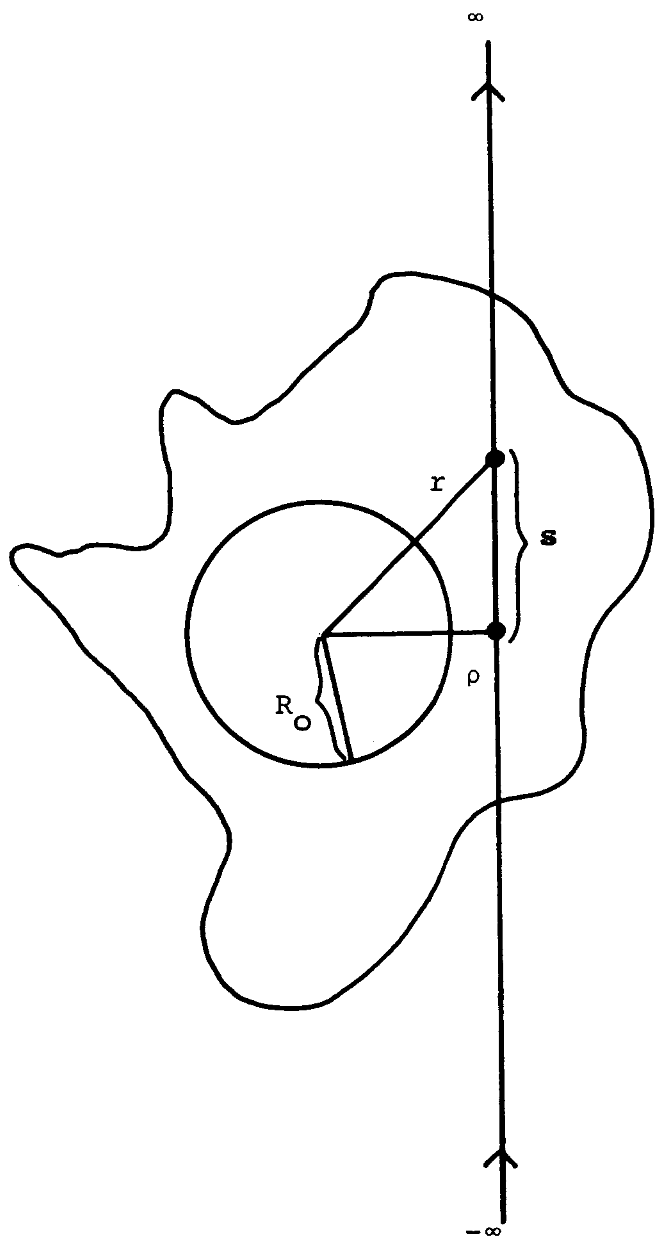
If f_1, f_2, and f_p are substituted into (3), we find that

$$T \simeq 2.13\cdot 10^{-19} N \text{ seconds}$$

where $N = \int n ds$ is the "columnar density", or total number of electrons per square centimeter along the transmission path. This columnar density, $N = N(\rho)$, is related to the radial density, $n(r)$, (see figure 1) by the integral

$$N(\rho) = \int_{-\infty}^{\infty} n(R)\,dS = R_o \int_{-\infty}^{\infty} n(r)\,ds \qquad (4)$$

where the lower case distances have been normalized with respect to the solar radius R_o. $N(\rho)$ will be found by measuring T for different points along the satellite trajectory, and hence for different ρ, the path offset. As seen above, the radial density $n(r)$ can then be obtained (Ref. 9, pps. 8-11). This



ρ - path offset

Figure 1 - Propagation Path Through Solar Corona

is the basis for the radio propagation experiment.

B. Communications Aspect of Sunblazer

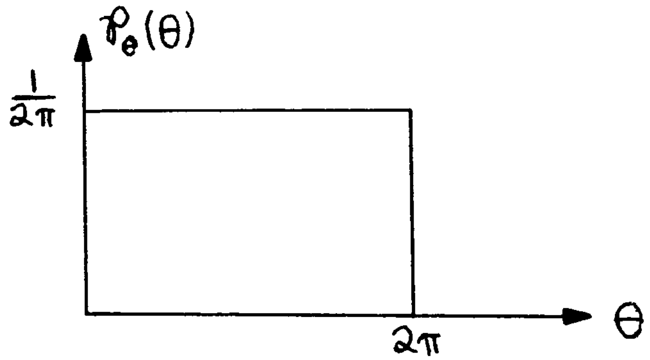
As seen above, we would like to measure as accurately as possible the arrival time of a received pulse. We can view this as a communication system which is modeled as in figure 2 (Ref. 5, p. 2). Note that this model neglects dispersion. The transmitted signal is

$$X(t) = \sqrt{2E_t}\, m(t)\cos\omega_o t = \mathrm{Re}[\sqrt{2E_t}\, m(t)e^{j\omega_0 t}] \qquad (5)$$

where m(t) is narrow-band and is normalized to unit energy i.e.

$$\int_{-\infty}^{\infty} m^2(t)dt = 1$$

The random phase, θ, is modeled by a uniform probability density as below



The attenuation is assumed to be known (random amplitude fluctuations are ignored). The propagation delay (or arrival time), τ_o, is treated as a real (non-random) but unknown parameter. Finally, the additive noise, n(t), is modeled as white Gaussian, with variance $N_o/2$. The received signal is then

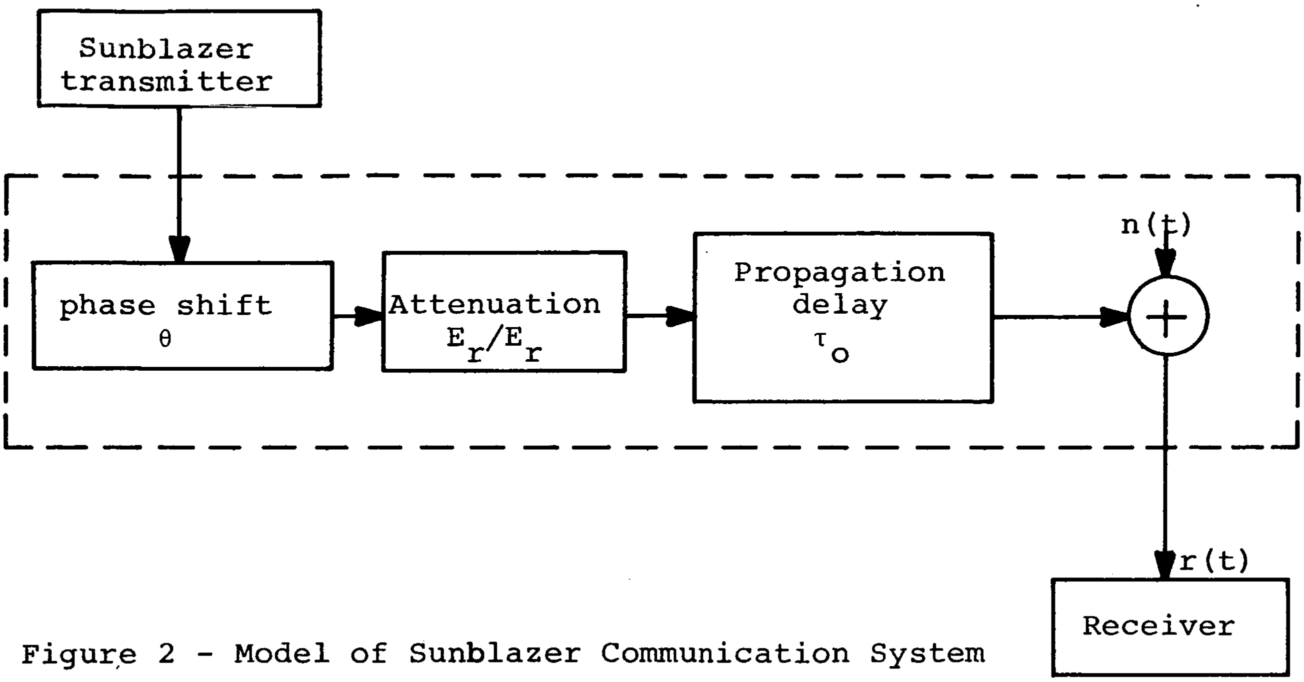


Figure 2 - Model of Sunblazer Communication System

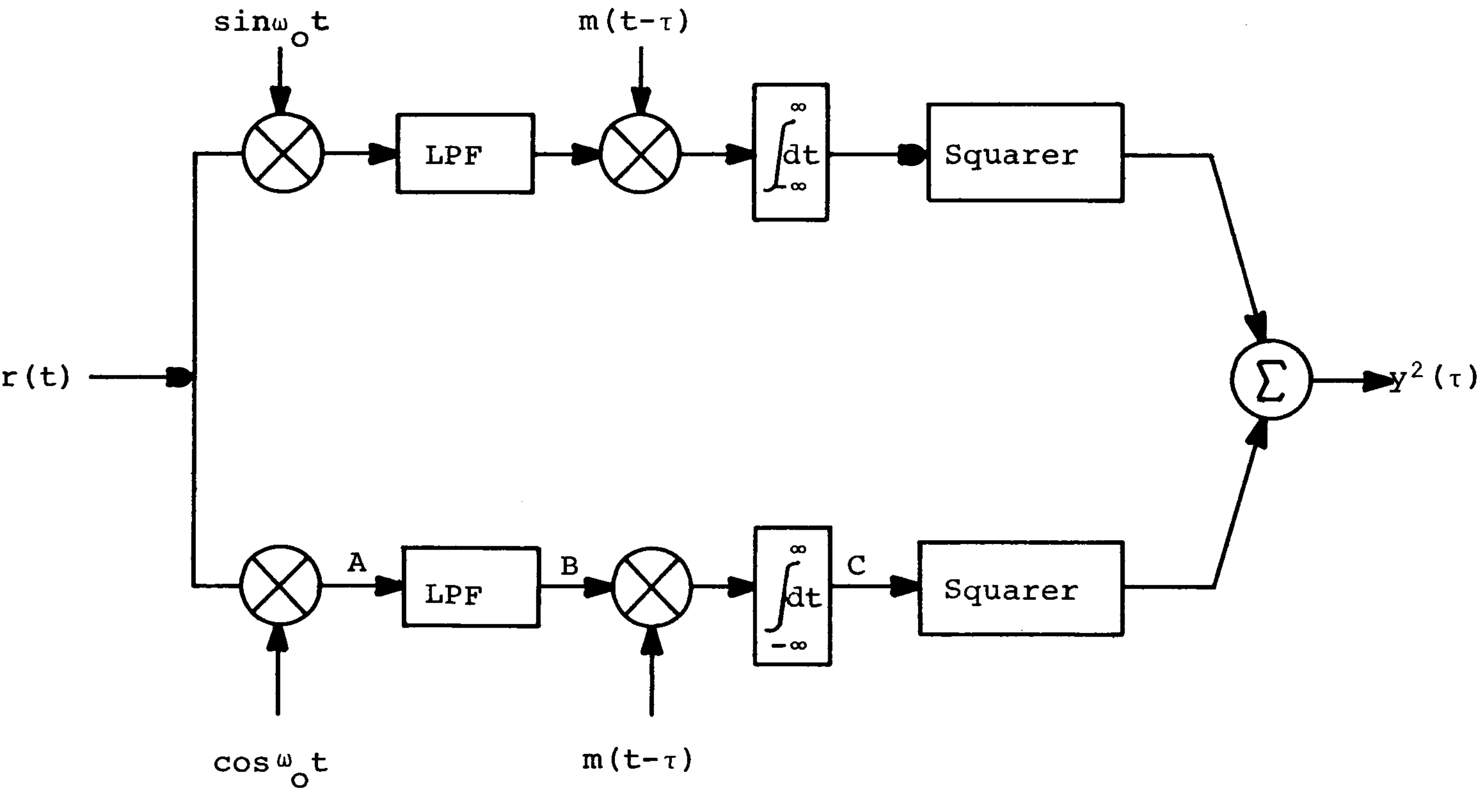


Figure 3 - The Correlation Receiver

$$r(t) = \sqrt{2E_r}\, m(t-\tau_o) \cos(\omega_o t + \theta) + n(t) \qquad (6)$$

There are two problems associated with this communication system. First, a decision must be made as to whether or not a signal is actually present; this is the detection problem. Then, assuming that a signal is present, the arrival time τ_o must be measured, this being the estimation problem. To achieve both detection and arrival time estimation for this phase incoherent situation, it is sufficient (and optimal) for the receiver to calculate the square of the correlation function of the demodulated signal (Ref. 17, Chap. 7). This "correlation receiver" is shown in figure 3. We can trace the operations performed on the received signal as it goes through the cosine channel as follows:

The received signal is

$$r(t) = \sqrt{2E_r}\, m(t-\tau_o)[\cos\omega_o t\cos\theta - \sin\omega_o t\sin\theta] + n(t)$$

Then:

$$\text{(A)}: \; r(t)\cos\omega_o t = \frac{\sqrt{2E_r}}{2} m(t-\tau_o)\cos\theta + \frac{\sqrt{2E_r}}{2} m(t-\tau_o)\cos 2\omega_o t\cos\theta$$

$$- \frac{\sqrt{2E_r}}{2} m(t-\tau_o)\sin 2\omega_o t\cos\theta + n(t)\cos\omega_o t$$

(B): Low-pass filter subtracts out high-frequency terms, leaving:

$$\frac{\sqrt{2E_r}}{2} m(t-\tau_o)\cos\theta + \tilde{n}_c(t)$$

where $\tilde{n}_c(t)$ is the low-frequency cosine component of the noise.

(C): After integration, we have

$$\frac{\sqrt{2E_r}}{2}\int_{-\infty}^{\infty} m(t-\tau_o)m(t-\tau)\ \cos\theta dt + \text{noise term}$$

We now introduce an important assumption about the random phase θ which will be part of our model of the channel. Namely, we assume that $\theta = \theta(t)$ is slowly-varying with respect to time. Specifically, θ must remain approximately constant over the period of integration. When this is not the case, the communication scheme will break down.

Then bringing $\cos\theta$ outside of the integral, we have

$$\frac{\sqrt{2E_r}}{2}\cos\theta\ R_{mm}(\tau-\tau_o) + \text{noise term}$$

where $R_{mm}(\tau-\tau_o)$ is the shifted auto-correlation function of the message waveform $m(t)$. Similarly, after integration in the sine channel we have

$$\frac{\sqrt{2E_r}}{2}\sin\theta R_{mm}(\tau-\tau_o) + \text{noise term}$$

(D): After squaring and adding we have, as the output of the optimum receiver,

$$y^2(\tau) = \frac{E_r}{2} R_{mm}^2(\tau-\tau_o) + \text{noise terms} \qquad (7)$$

A decision as to the presence of the signal (detection) is then made based on the height of the central peak of this output $y^2(\tau)$, and an estimate of τ_o is made by looking at the time-position of the central peak. Some measures of performance can then be chosen that will tell us how certain our detection decision is and how accurately we have estimated τ_o. Since the receiver is optimal (in the mathematical sense), we will then know what is the best possible performance that we can expect. The details of how this is done are left for the next chapter, and at this point it is important only to be aware of two points.

First, detection performance depends only on the peak value of R_{mm}, and hence only on the signal energy. Thus no amount of coding can improve detection performance. Coding can, however, be beneficial with respect to arrival time estimation. It turns out (and will be derived in the next chapter) that for large signal-to-noise ratio the mean-square estimation error is given by

$$\overline{\varepsilon^2(\tau)} = \left(\frac{1}{8\pi^2}\right) \frac{1}{(E_r/N_o)W^2}$$

where W is the effective bandwidth of the signal. What we would like to do then, is for a given available energy, transmit a signal with as large a bandwidth as possible. Theoretically we could put all of the energy into a single narrow pulse. This is impossible, however, because of the practical problem of peak power limitations that arises in building the transmitters. It is necessary, then, to take a long, low-power pulse, and code it so as to increase its bandwidth. A class of codes which achieve this are the Barker codes.

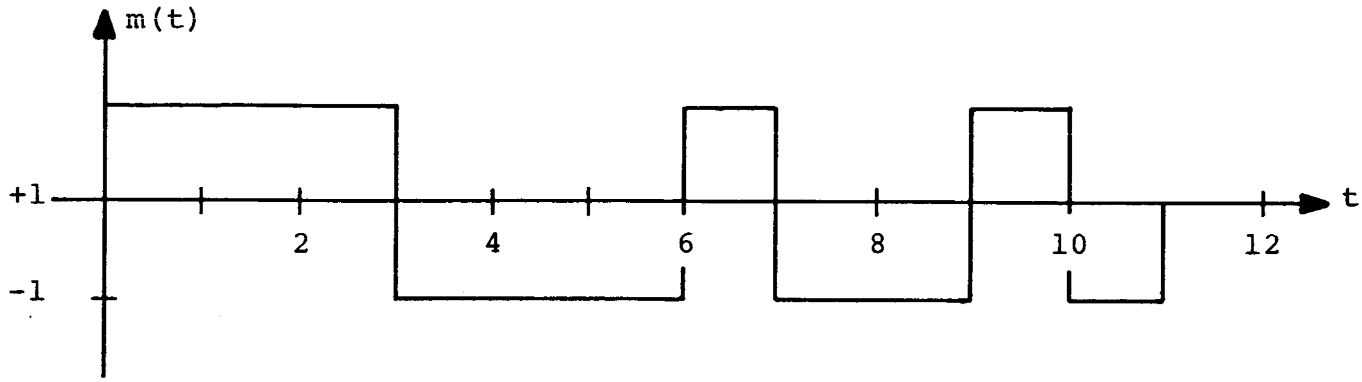


Figure 4a - Eleven-bit Barker Code

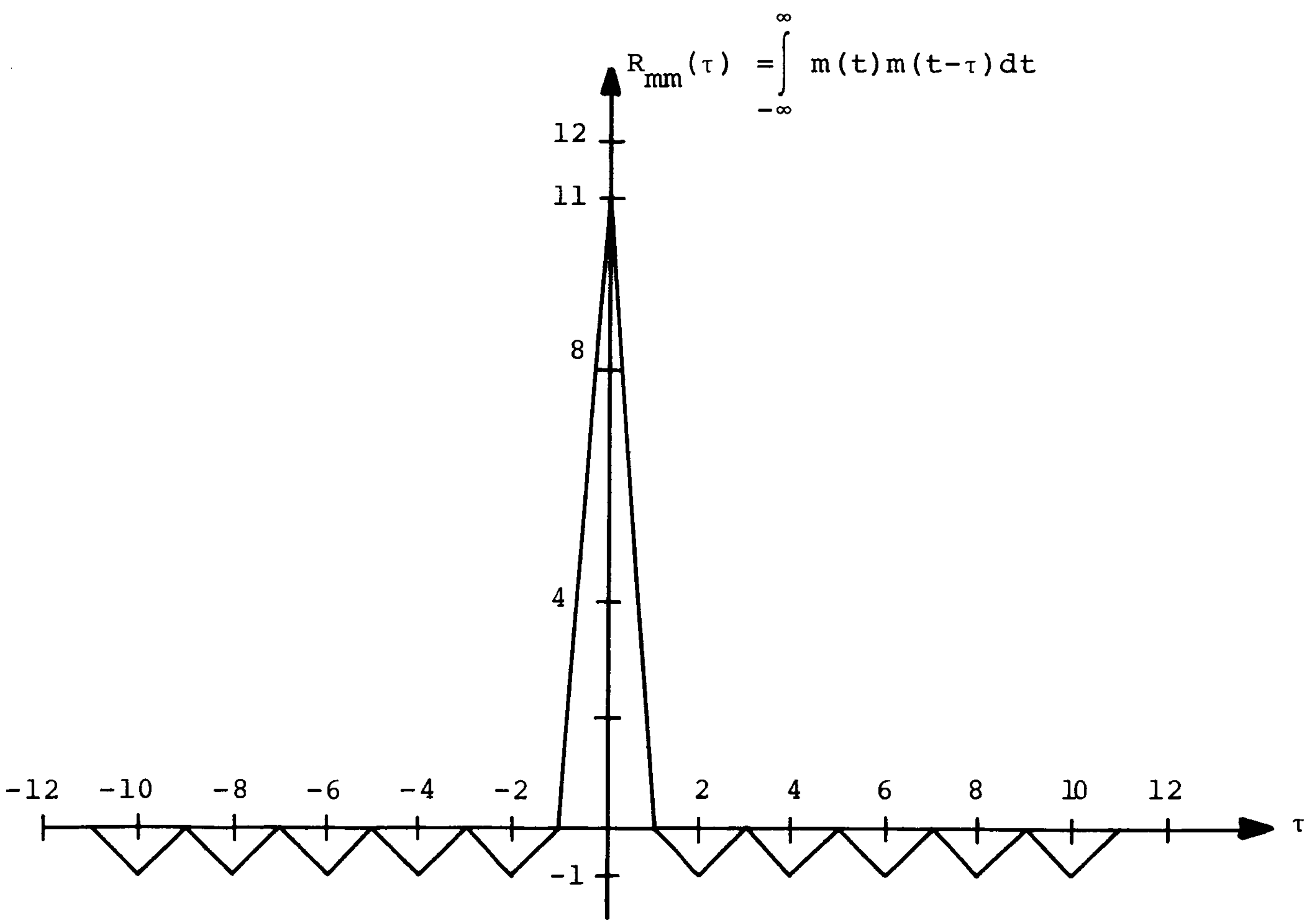


Figure 4b - Corresponding Auto-correlation Function

These are pseudo-random codes whose auto correlation functions have a central peak N times as high as any of the side peaks, where N is the number of bits in the code. The eleven-bit Barker code, with its auto-correlation function, is shown in figure 4. Although the longest known Barker code is thirteen bits long, the basic code can be folded on itself to increase the energy. Sunblazer will probably use a triply-folded version of the eleven-bit code (Ref. 9 pps. 46-48).

C. Dispersion

The reception scheme that has just been described does not take into account the dispersive effect of the plasma. In fact, the received signal will be a corrupted version of the signal that was expected when the optimum receiver was designed. The performance of the receiver will be degraded--the estimation error will become larger, and more important, if the dispersion becomes large enough it may not even be possible to detect the signal. Also, it is difficult to compensate for the effect before hand by changing the receiver design, since the degree of dispersion will change with time and will often be unpredictable.

Clearly dispersion might prove to be an undesirable problem in the practical sense, and this would be reason enough to examine it in detail. The problem also has import in the theoretical sense, as becomes apparent if we consider the following points.

First, one should realize that the primary Sunblazer experiment depends upon the fact that the medium is dispersive. Dispersion simply implies that the phase velocity of a propagating wave is dependent on frequency. If the phase velocity was not a function of frequency, then the group velocity of the signal could not be a function of frequency either. Hence dispersion doesn't enter into the picture as simply an annoying side effect; it is an inherently necessary phenomenon.

Although dispersion is a necessary effect, the degree to which it manifests itself is perhaps not as clear. If our signal was to be a long pulse (essentially monochromatic) it would be virtually unaffected by dispersion. However, the accuracy with which we can estimate the arrival time of the signal becomes better as we make our signal bandwidth larger. This, of course, is why we code the signal in the first place. But the larger we make the bandwidth the more significant becomes the effect of dispersion. And, as will be shown later, as the dispersion increases the estimation error also increases. What we have, then, is a trade-off. On the one hand we know that increasing the bandwidth decreases the estimation error, but on the other hand the degree of dispersion becomes larger which increases the estimation error. The point here is that dispersion puts an inherent mathematical limitation on the best performance that we can hope to achieve. In this sense dispersion should naturally be part of a communication theory approach to the Sunblazer type problem.

D. Thesis Objectives

The primary goal of this thesis is to see how dispersion fits into the communication problem, and what it does to receiver performance. To do this I have first defined and examined some measures of receiver performance, for both detection and estimation. Then I have expanded and generalized these measures so that they would be meaningful when applied to a corrupted signal that has been processed by the correlation receiver described earlier. Next we will find a way to see just what the dispersed signal looks like, and determine a meaningful measure of dispersion. This is then applied to the specific case of the coronal plasma, and the measure of dispersion is evaluated for different points in the Sunblazer trajectory. Finally numerical results are obtained that tell us exactly what happens to the receiver performance as the amount of dispersion changes.

II. THE COMMUNICATION PROBLEM

In this chapter we will introduce dispersion into our model for the communication system, and see how it is related to receiver performance. As a primary result, we will find that to determine the change in receiver performance it is necessary and sufficient to calculate the cross-correlation function of the dispersed and un-dispersed signal. In arriving at this we will first examine in some detail the detection and estimation problems for the case of no dispersion, and in doing so we will introduce some measures of receiver performance. It will then be easy to re-evaluate these performance measures for the case of a dispersed signal.

A. Detection - No Dispersion

We have seen that $y^2(\tau)$, the output of the correlation receiver, consists of a term proportional to the square of the auto-correlation function of m(t), plus added noise terms. This output can be observed and used to decide which one of two hypotheses we believe is correct--namely, a signal is present, or a signal is not present. We would expect to have a decision rule of the form: "A signal is present if the output of the correlation receiver (peak output) is greater than some threshold value, say ξ." In figure 5 a typical example is shown in which we would say that a signal is present.

ROC Curves As a Performance Measure:

We should first recognize that there are to possible types of detection error that can occur, which we can call "miss" and "false alarm". In the case of a miss, our peak

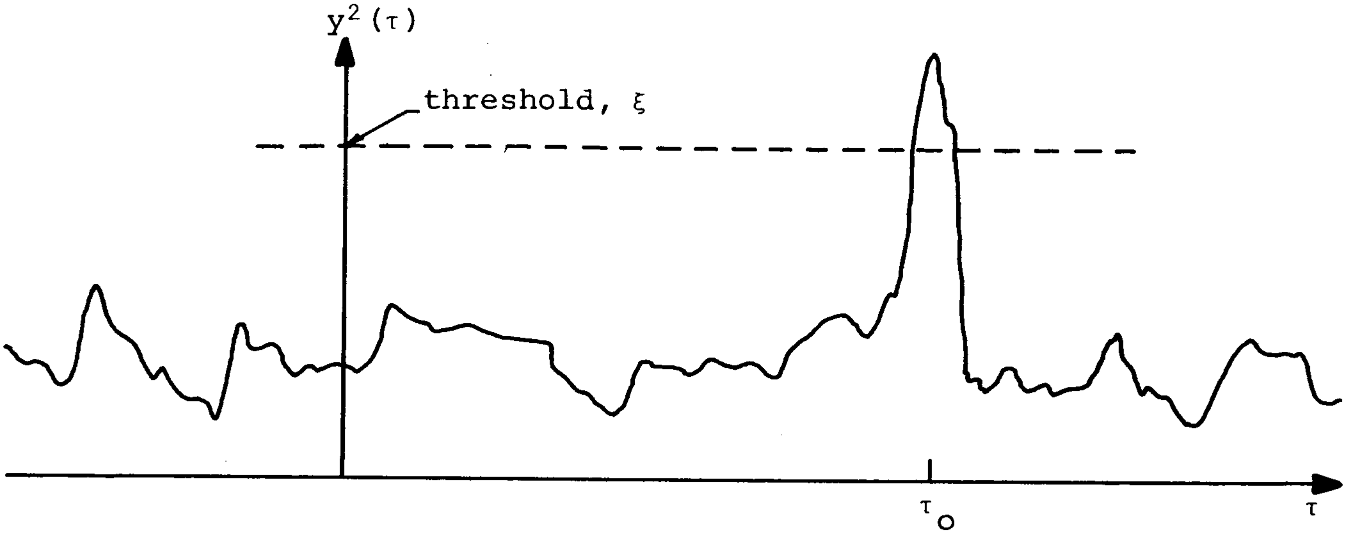


Figure 5 - Typical Output of Correlation Receiver

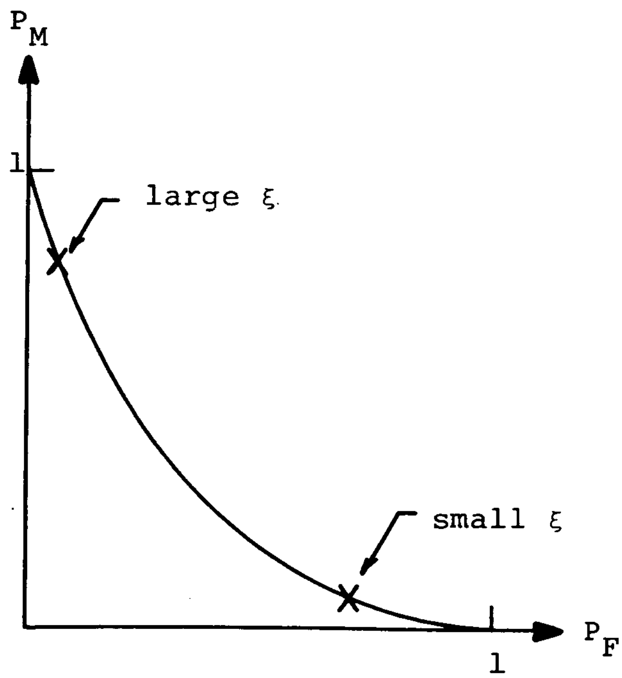


Figure 6a - Curve Relating P_M and P_F

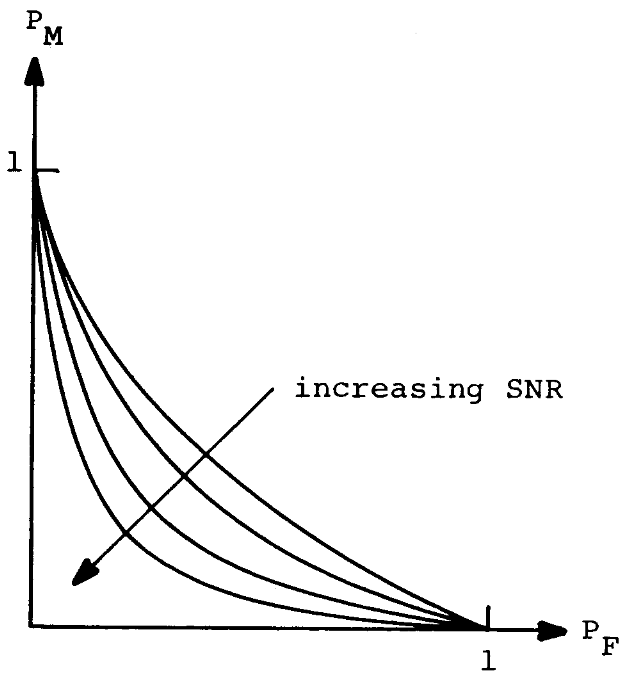


Figure 6b - Complete ROC

correlation output is below the chosen threshold ξ, even though a signal is actually present. In the case of a falsealarm, a peak is above the threshold though no signal is present. The associated error probabilities are then

$$P_M = P_{miss} = \Pr[\mathcal{E} \mid \text{signal present}]$$
$$P_F = P_{false\ alarm} = \Pr[\mathcal{E} \mid \text{signal absent}]$$
$$P_{miss} = 1 - P_{detection}$$

For a given signal-to-noise ratio[1] we will have a trade-off between P_M and P_F, and we could draw a curve relating the two (figure 6a). Where we are on this curve depends on our choice of threshold, ξ. For example, choosing ξ large will result in a small chance of false alarm, but a large change of miss. We can make P_M and P_F whatever we want as long as we limit ourselves to the curve; to get off of the curve we would have to change the SNR. In general, we could calculate an entire family of curves for different SNR that would completely describe the operation of the receiver as a detector (figure 6b). This set of curves is called the "receiver operating characteristic" (ROC).

The point here is that our choice of ξ, i.e. our decision rule, is somewhat arbitrary in that it depends on what type of performance we desire. For example, we might assign a certain cost to making each type of error, and try to minimize the average cost:

[1]SNR = E_r/N_o = signal energy over noise spectral density.

$$\overline{C} = C(M)P_M + C(F)P_F$$

Here we would pick ξ so as to minimize $\overline{C}$. This average cost would then be our performance measure.

We could also use a total error probability for a performance measure, namely:

$$P(\varepsilon)=\Pr\ [\text{signal present}]\cdot P_M+\Pr\ [\text{signal absent}]\cdot P_F$$

This total $P(\varepsilon)$, however, is often not very meaningful since the a priori probabilities are usually either unknown or are changing with time. The average cost $\overline{C}$ is also of questionable value as a performance measure, since values for C(M) and C(F) are difficult to assign and may also change with time.

I feel that the set of ROC curves alone is the best way of describing receiver performance, particularly for Sunblazer. From these curves the threshold ξ can be determined for any decision criterion (e.g. minimax, Neyman-Pearson, etc.) and a corresponding performance measure can be evaluated.

Calculation of the ROC Curves

We are interested in the statistics of the peak value of $y(\tau)$, the output statistic of the correlation receiver. Suppose that we could write the conditional probability densities for this peak output (call it y) given that a signal is absent, and given that a signal is present (figure 7). P_M and P_F could then be found by integrating the densities over the regions indicated. For a given SNR we would then

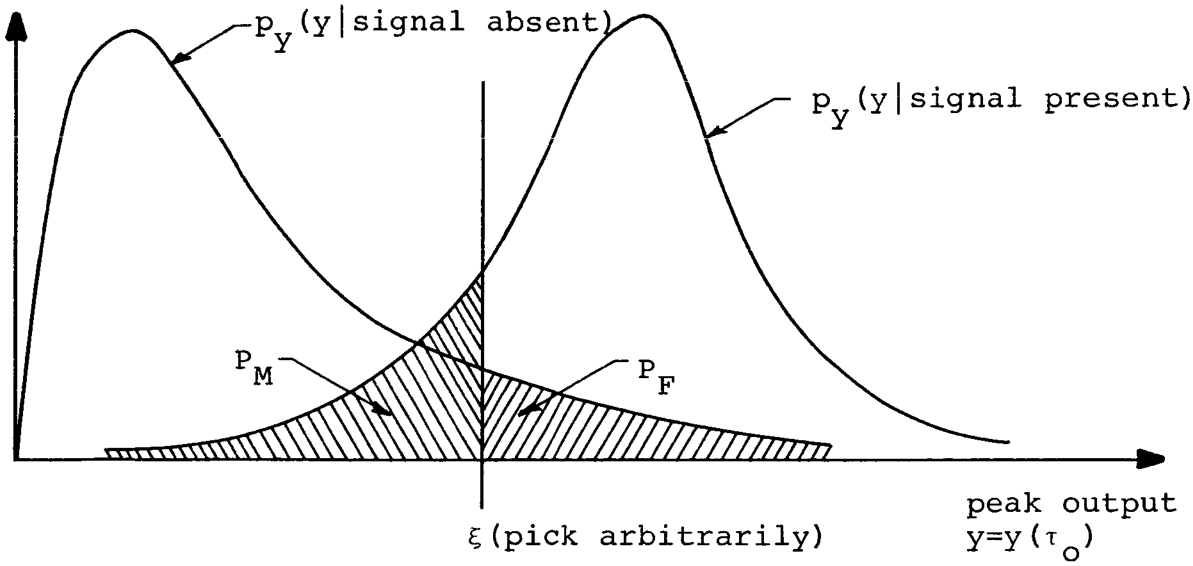


Figure 7 - Conditional Densities for Peak Output

have P_M and P_F both as a function of ξ and obtain P_M in terms of P_F, which is the desired result.

For the case of white Gaussian noise and coherent phase the solution of this problem is not too bad, and it is possible to determine an analytic expression for P_M as a function of P_F (Ref. 11). However, for our case of incoherent phase the probability densities contain Bessel functions, and the solution is considerably more difficult. In fact an analytic solution for P_M in terms of P_F is impossible, and elimination of ξ must be carried out by means of machine computation. Interestingly, it turns out that if random amplitude is included in the channel model (Rayleigh density) together with incoherent phase, an analytic solution is possible. It is not my intention, however, to compute the ROC curves for any of these cases. The problem is a classical one, and appears in the references (Ref. 11, chap. 3; Ref. 16, chap. 2).

B. Estimation - No Dispersion

In estimating the signal's arrival time we must again consider two types of errors that can occur. The first type, which we can call "catastrophic error", results from the presence of secondary peaks on the function $y^2(\tau)$ which are removed from the main peak. If the receiver should lock on to one of these secondary peaks, a large estimation error will, of course, occur. For large signal-to-noise ratio there is little chance of this happening. If the SNR was low enough to make catastrophic error a significant problem, it would also be too low to successfully detect the signal. In this case, present Sunblazer plans call for sequential detection, which effectively raises the energy, and makes catastrophic error again improbable. For this reason I have chosen to not analyze this type of error in any detail.

The second type of error occurs when noise causes slight shifts in the position of the main peak. This type of error is, of course, always present, and results in a small inaccuracy in the estimation of arrival time. A good performance measure associated with this kind of error is the mean square estimation error, $\overline{\varepsilon^2(\tau)} = \overline{(\hat{\tau}-\tau_o)^2}$, where $\hat{\tau}$ is our estimate of the actual arrival time τ_o. This mean square error is dependent not only on the signal-to-noise ratio, but also on the signal bandwidth. This dependence can be derived as follows.

Assume the transmitted signal is of the form

$$S(t) = m(t)\cos(\omega_o t-\theta)$$

$$\text{with} \int_{-\infty}^{\infty} m^2(t)\,dt = E_t = E_r$$

Here, for simplicity, we have assumed that attenuation and random phase are introduced at the transmitter. The received signal is then

$$r(t) = S(t-\tau_o) + n(t)$$

where τ_o is the actual arrival time. Let $\hat{\tau}$ be our estimate of τ_o. Then we would like our receiver to pick $\hat{\tau}=\tau$ such that the density $p[\tau|r(t)]$ is maximized.

$$p[\tau|r(t)] = \frac{p[\tau,r(t)]}{p[r(t)]} = \frac{p[r(t)|\tau]p(\tau)}{p[r(t)]} \tag{8}$$

$p[r(t)]$ is independent of τ, so we can equivalently maximize $p[r(t)|\tau]\cdot p(\tau)$. Then, assuming no a priori knowledge of τ, we are left with maximizing $p[r(t)|\tau]$ (MAP estimate).

$$p_{r(t)|\tau}[r(t)|\tau] = p_{n(t)}[r(t)-S(t)|\tau,\theta] \tag{9}$$

Average this over the random phase θ:

$$p_{n(t)}[r(t)-S(t)|\tau] = \overline{p_{n(t)}[r(t)-S(t)|\tau,\theta]}^{\theta}$$

$$= \overline{p_{n(t)}[r(t)-S(t-\tau)|\theta]}^{\theta} \tag{10}$$

The noise n(t) is a Gaussian random process that can be projected onto a set of orthornormal functions to yield independent Gaussian random variables.

$$p_{\underline{n}}(\underline{\mu}) = \frac{1}{(\pi N_o)^{N/2}} e^{-\frac{1}{N_o}|\underline{\mu}|^2}$$

where N is the number of orthonormal functions in the expansion.

$$|\underline{\mu}|^2 = \underline{\mu}\cdot\underline{\mu} = \int_{\infty}^{\infty} \mu^2(t)\,dt$$

then, $$p_{n(t)}[\mu(t)] = \frac{1}{\sqrt{\pi N_o}} e^{-\frac{1}{N_o}\int_{-\infty}^{\infty}\mu^2(t)\,dt} \tag{11}$$

$$\overline{p_{n(t)}[r(t)-S(t-\tau)|\theta]}^{\theta} = \frac{1}{\sqrt{\pi N_o}} \overline{e^{-\frac{1}{N_o}\int_{-\infty}^{\infty}[r(t)-S(t-\tau)]^2 dt}}^{\theta} \tag{12}$$

$$\int_{-\infty}^{\infty}[r(t)-S(t-\tau)]^2 dt = \int_{-\infty}^{\infty}[r^2(t)-2r(t)S(t-\tau) + S^2(t-\tau)]\,dt$$

The integrations of $r^2(t)$ and $S^2(t-\tau)$ will not depend on τ; therefore they can be ignored and replaced by a constant. The receiver, then, should maximize:

$$K\overline{\epsilon^{\frac{2}{N_o}\int_{-\infty}^{\infty} r(t)\ S(t-\tau)\,dt}}^{\theta}$$

$$= K\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{2}{N_o}\int_{-\infty}^{\infty} r(t)m(t-\tau)\cos(\omega_o t-\theta)\,dt}\,d\theta \tag{13}$$

since $\cos(\omega_o t-\theta) = \cos\omega_o t\cos\theta + \sin\omega_o t\sin\theta$, we have:

$$P_{n(t)}[r(t) - S(t-\tau)] =$$

$$K\cdot\frac{1}{2\pi}\int_{0}^{2\pi} e^{\frac{2}{N_o}[\cos\theta\int_{-\infty}^{\infty} m(t-\tau)r(t)\cos\omega_o t\,dt + \sin\theta\int_{-\infty}^{\infty} m(t-\tau)r(t)\sin\omega_o t\,dt]}\,d\theta$$

$$= K\cdot I_o\left(\frac{2}{N_o}\sqrt{L_c^2 + L_s^2}\right) \tag{14}$$

Where $I_o(\ )$ is the zero-order modified Bessel function,

$$\text{and } L_c(\tau) = \int_{-\infty}^{\infty} m(t-\tau)r(t)\cos\omega_o t\ dt \tag{15a}$$

$$\text{and } L_s(\tau) = \int_{-\infty}^{\infty} m(t-\tau)r(t)\sin\omega_o t\ dt \tag{15b}$$

But we observe that $L_c^2(\tau) + L_s^2(\tau)$ is just the output of the correlation receiver (the function of the low-pass filtering is carried out, in effect, by the integration).

$$\int_{-\infty}^{\infty} m(t-\tau)r(t)\cos\omega_o t\ dt = \int_{-\infty}^{\infty} m(t-\tau)[m(t-\tau_o) + n_c(t)]dt \tag{16}$$

Since $n_c(t)$ is zero mean, and $n(t)$ and $m(t)$ are uncorrelated, the integral of $m(t-\tau)n_c(t)$ will be approximately zero, especially compared to $R_{mm}(\tau-\tau_o)$. Hence we disregard it, and are left with:

$$p_{n(t)}[r(t)-S(t-\tau)] = K\cdot I_o\left(\frac{2}{N_o}|R_{mm}(\tau-\tau_o)|\right) \tag{17}$$

Approximation of Large Signal-to-Noise Ratio:

If $\frac{|R_{mm}(\tau-\tau_o)|}{N_o}$ is large, we can approximate $I_o(X)$ by its asymtotic value for large X (Ref. 1):

$$I_o(X) \sim \frac{e^X}{\sqrt{X}}$$

$$\text{or}\quad p_{n(t)}[r(t)-S(t-\tau)] \sim \frac{1}{\sqrt{\frac{2}{N_o}|R_{mm}(\tau-\tau_o)|}}\ e^{\frac{2}{N_o}|R_{mm}(\tau-\tau_o)|} \tag{18}$$

We will be interested in the variance of this density--i.e. how fast it varies about τ_o. But $\sqrt{|R_{mm}(\tau-\tau_o)|}$ is slowly varying comapred to $e^{|R_{mm}(\tau-\tau_o)|}$, which is very highly peaked at $\tau=\tau_o$. Hence we ignore the $\sqrt{|R_{mm}(\tau-\tau_o)|}$ term, and write that:

$$p_{n(t)}[r(t)-S(t-\tau)] \sim e^{\frac{2}{N_o}|R_{mm}(\tau-\tau_o)|} \tag{19}$$

We now expand $|R_{mm}(\tau-\tau_o)|$ in a Taylor series about $\tau = \tau_o$:

$$R_{mm}(\tau-\tau_o) \simeq R_{mm}(0) + \frac{d}{dt}|R_{mm}(\tau-\tau_o)|\Big|_{\tau=\tau_o} \cdot (\tau-\tau_o) + \frac{d^2}{d\tau^2}|R_{mm}(\tau-\tau_o)|\Big|_{\tau=\tau_o} \cdot \frac{(\tau-\tau_o)^2}{2}$$

Since $R_{mm}(\tau-\tau_o)$ is a maximum at $\tau = \tau_o$, the first derivative term is zero. Also, $R_{mm}(0)$ is not a function of τ and hence can be ignored. We have, then,

$$p_{n(t)}[r(t)-S(t-\tau)] \sim e^{\frac{1}{N_o}|R_{mm}(\tau-\tau_o)|''\Big|_{\tau=0} \cdot(\Delta\tau)^2} \tag{20}$$

where $\Delta\tau = \tau-\tau_o$

This is the probability density for the estimation error, $\Delta\tau$. Then $\overline{(\Delta\tau)^2} = \overline{\varepsilon^2(\tau)}$ = the variance of this density (since it is a Gaussian).

$$\overline{\varepsilon^2(\tau)} = \frac{-1}{\frac{2}{N_o}|R_{mm}(\tau)|''\Big|_{\tau=0}}$$

$$|R_{mm}(\tau)| = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega\tau}\,\Phi_{mm}(\omega)\,d\omega$$

where $\Phi_{mm}(\omega)$ is the spectral density of $|R_{mm}(\tau)|$.

$$|R_{mm}(\tau)|'' = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2\Phi_{mm}(\omega)e^{j\omega\tau}d\omega$$

$$|R_{mm}(0)|'' = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2\Phi_{mm}(\omega)d\omega$$

$$= -E_r W^2$$

Here W is the "effective", or mean-square, bandwidth, defined by:

$$W^2 = \frac{\int_{-\infty}^{\infty}\omega^2\Phi_{mm}(\omega)d\omega}{\int_{-\infty}^{\infty}\Phi_{mm}(\omega)d\omega} = \frac{1}{2\pi E_r}\int_{-\infty}^{\infty}\omega^2\Phi_{mm}(\omega)\, d\omega \tag{21}$$

We have, then, <u>for large SNR</u>:

$$\overline{\varepsilon^2(\tau)} = \frac{1}{2(E_r/N_o)W^2} \tag{22a}$$

for W in radians per second, or

$$\overline{\varepsilon^2(\tau)} = \frac{1}{8\pi^2(E_r/N_o)W^2} \tag{22b}$$

for W in cycles per second.

<u>Approximation of Small Signal-to-Noise Ratio</u>:

If $|R_{mm}(\tau-\tau_o)|/N_o$ is small, we can approximate the modified Bessel function by the first two terms in its power series representation:

$$I_o(X) = 1 + \frac{\frac{1}{4}X^2}{(1!)^2} + \frac{(\frac{1}{4}X^2)^2}{(2!)^2} + \frac{(\frac{1}{4}X^2)^3}{(3!)^2} + \ldots.$$

$$\simeq 1 + \frac{1}{4}X^2$$

$$\ln p_{n(t)}[r(t)-S(t-\tau)] = \ln K + \ln[1 + \frac{1}{N_o^2} R_{mm}^2(\tau-\tau_o)]$$

for small y, $\ln(1 + y) \simeq y$, so we have:

$$\ln p_{n(t)}[r(t)-S(t-\tau)] \simeq \ln K + \frac{1}{N_o^2} R_{mm}^2(\tau-\tau_o)$$

or, $p_{n(t)}[r(t)-S(t-\tau)] = K'e^{\frac{1}{N_o^2} R_{mm}^2(\tau-\tau_o)}$ (23)

Now expand $R_{mm}^2(\tau-\tau_o)$ in a Taylor series about τ_o:

$$R_{mm}^2(\tau-\tau_o) \simeq A + \frac{1}{2} \frac{d^2}{d\tau^2} R_{mm}^2(\tau-\tau_o)\Big|_{\tau=\tau_o} \cdot (\Delta\tau)^2$$

Then, $p_{n(t)}[r(t)-S(t-\tau)] \simeq e^{\frac{1}{2N_o^2}(R_{mm}^2(\tau))''\big|_o \cdot (\Delta\tau)^2}$ (24)

and $\overline{(\Delta\tau)^2} = \overline{\varepsilon^2(\tau)} = \dfrac{-1}{(R_{mm}^2(\tau))''\big|_o \cdot \frac{1}{N_o^2}}$ (25)

$$\frac{d^2}{d\tau^2}(R_{mm}^2(\tau)) = 2\left(\frac{d|R_{mm}(\tau)|}{d\tau}\right)^2 + 2|R_{mm}(\tau)| \cdot \frac{d^2|R_{mm}(\tau)|}{d\tau^2}$$

The first term on the right is zero, so:

$$\overline{\varepsilon^2(\tau)} = \frac{-1}{\frac{2}{N_o^2}\left|R_{mm}(0)\right| \cdot \left|R_{mm}(0)\right|''} \quad (26)$$

$$|R_{mm}(0)| = E_r$$

$$|R_{mm}(0)|'' = -E_r W^2$$

We have, then, <u>for small SNR</u>:

$$\overline{\varepsilon^2(\tau)} = \frac{1}{2(E_r/N_o)^2 W^2} \quad (27a)$$

For W in radians per second, or

$$\overline{\varepsilon^2(\tau)} = \frac{1}{8\pi^2(E_r/N_o)^2W^2} \tag{27b}$$

for W in cycles per second.

An important point in these derivations is that $\overline{\varepsilon^2(\tau)}$ depends only on the magnitude of the autocorrelation function, $|R_{mm}(\tau)|$ or $R^2_{mm}(\tau)$, which is the output of the correlation receiver. This came from the integrals $L_c(\tau)$ and $L_s(\tau)$ in equations 14 and 15. Suppose now that the received signal consists of noise plus a corrupted version of m(t), say a(t) (it doesn't really matter what kind of corruption it is). We will show next that if we put this corrupted signal into our correlation receiver, the output will be the squared cross-correlation function $R^2_{am}(\tau)$. The mean square error of arrival time estimation will depend only on this cross-correlation function, in particular, on its maximum value and its mean-square bandwidth.

C. Detection and Estimation for the Dispersed Case.

Suppose now that our received signal is of the form

$$r(t) = \mathrm{Re}[a(t-\tau_o)e^{j(\omega_o t+\theta)}] + n(t) \tag{28}$$

where a(t), the dispersed version of m(t), can be complex,

i.e. $a(t) = a_r(t) + ja_i(t)$

We want to see what the output of the correlation receiver is. Referring to the cosine channel in figure 2:

A: $r(t)\ \cos\omega_o t = \mathrm{Re}[a(t-\tau_o)e^{j(\omega_o t+\theta)}]\cos\omega_o t$

$$+ n(t)\cos\omega_o t$$

$$=a_r(t-\tau_o)\cos(\omega_o t+\theta)\cos\omega_o t-a_i(t-\tau_o)\sin(\omega_o t+\theta)\cos\omega_o t+n(t)\cos\omega_o t$$

$$=a_r(t-\tau_o)[\cos\omega_o t\cos\theta-\sin\omega_o t\sin\theta]\cos\omega_o t$$

$$-a_i(t-\tau_o)[\sin\omega_o t\cos + \cos\omega_o t\sin\theta]\cos\omega_o t + n(t)\cos\omega_o t$$

$$=a_r(t-\tau_o)[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\omega_o t]\cos\theta-a_r(t-\tau_o)\cdot\tfrac{1}{2}\sin 2\omega_o t\sin\theta$$

$$-a_i(t-\tau_o)\tfrac{1}{2}\sin 2\omega_o t\cos -a_i(t-\tau_o)[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\omega_o t]\sin\theta + n(t)\cos\omega_o t$$

B: After deleting the high-frequency terms, we are left with:

$$\tfrac{1}{2}a_r(t-\tau_o)\cos\theta - \tfrac{1}{2}a_i(t-\tau_o)\sin\theta + \tilde{n}_c(t)$$

C: After integration, we have:

$$\tfrac{1}{2}\cos\theta R_{a_r m}(\tau-\tau_o)- \tfrac{1}{2}\sin\theta R_{a_i m}(\tau-\tau_o) + \text{noise terms}$$

Similarly, after integration on the sine channel we have:

$$- \tfrac{1}{2}\sin\ R_{a_r m}(\tau-\tau_o)- \tfrac{1}{2}\cos\theta R_{a_i m}(\tau-\tau_o) + \text{noise terms}$$

D: Squaring and adding, and ignoring the noise terms, we have:

$$y^2(\tau) = \tfrac{1}{4}\cos^2\theta R^2_{a_r m}(\tau-\tau_o) + \tfrac{1}{4}\sin^2\theta R^2_{a_i m}(\tau-\tau_o)$$

$$-\sin\theta\cos\theta R_{a_r m}(\tau-\tau_o)R_{a_i m}(\tau-\tau_o)$$

$$+ \tfrac{1}{4}\sin^2\theta R^2_{a_r m}(\tau-\tau_o)+\tfrac{1}{4}\cos^2\theta R_{a_i m}(\tau-\tau_o)+\sin\theta\cos\theta R_{a_r m}(\tau-\tau_o)R_{a_i m}(\tau-\tau_o)$$

or, $y^2(\tau) = \tfrac{1}{4}\ R^2_{a_r m}(\tau-\tau_o)+\tfrac{1}{4}R^2_{a_i m}(\tau-\tau_o)$

$$y^2(\tau) = \frac{1}{4}R^*_{am}(\tau-\tau_o)R_{am}(\tau-\tau_o) = \frac{1}{4}|R_{am}(\tau-\tau_o)|^2 \qquad (29)$$

The output of the correlation receiver, then, consists of the squared magnitude of the cross-correlation function of the dispersed and undispersed modulation, buried, again, in noise (for simplicity we have set $\sqrt{2E_r} = 1$). We would now like to use this fact to examine the problem of detecting and estimating dispersed signals.

Detection:

We have seen before that the most general way of describing the receiver as a detector is to calculate its set of ROC curves. This, in turn, depends on the statistics of the peak correlation output. Specifically, we had to know the two conditional probability densities for the random variable y, given a signal present or absent. Then to determine the ROC curves (and thus any other performance measure) for the dispersed case, it is necessary to recalculate the two conditional densities. The density $p_y(y|\text{signal absent})$, and its corresponding probability P_F, will not be any different since if there is no signal present dispersion is irrelevant.

If the density $p_y(y|\text{signal present})$ is known for the undispersed case, recalculating it for the dispersed case is also simple. This density is generally a function of both the random variable y and the signal energy E_r. Dispersion, however, is an energy-conserving process; the shape of the signal may change, but its total energy always remains constant. Rewriting the density, then, requires simply a transformation of the variable y.

If we let y_d be the peak output for a dispersed signal, and plot it as a function of increasing dispersion, we might expect a curve (ignoring noise) such as the one drawn in figure 8. We could say, then, that $y_d = \alpha y$, where α is a function of the amount of dispersion, and depends on the particular signal. Then to get the new density $p_{y_d}(y_d|\text{signal present})$, we make the transformation $y_d = \alpha y$ and get:

$$p_{y_d}(y_d|\text{signal present}) = \frac{1}{\alpha}\, p_y(\frac{y_d}{\alpha}|\text{signal present}) \qquad (30)$$

This density could then be integrated to get the new miss probability (again dependent on the threshold), P_{M_d}, which would be a function of α. From this, and the P_F calculated previously, a new set of ROC curves could be found for any particular α.

Estimation:

We would now like to rewrite the expression for the mean square estimation error for the dispersed case. $L_c^2(\tau) + L_s^2(\tau)$ (eq. 15) is again the output statistic of the correlation receiver. It now, of course, represents the cross-correlation function of the dispersed and undispersed signals:

$$L_c^2(\tau) + L_s^2(\tau) = |R_{am}(\tau-\tau_o)|^2 + \text{noise terms}$$

As long as $R_{am}(\tau-\tau_o)$ is still peaked at $\tau=\tau_o$, we can use the same analysis as before. For large signal-to-noise ratio

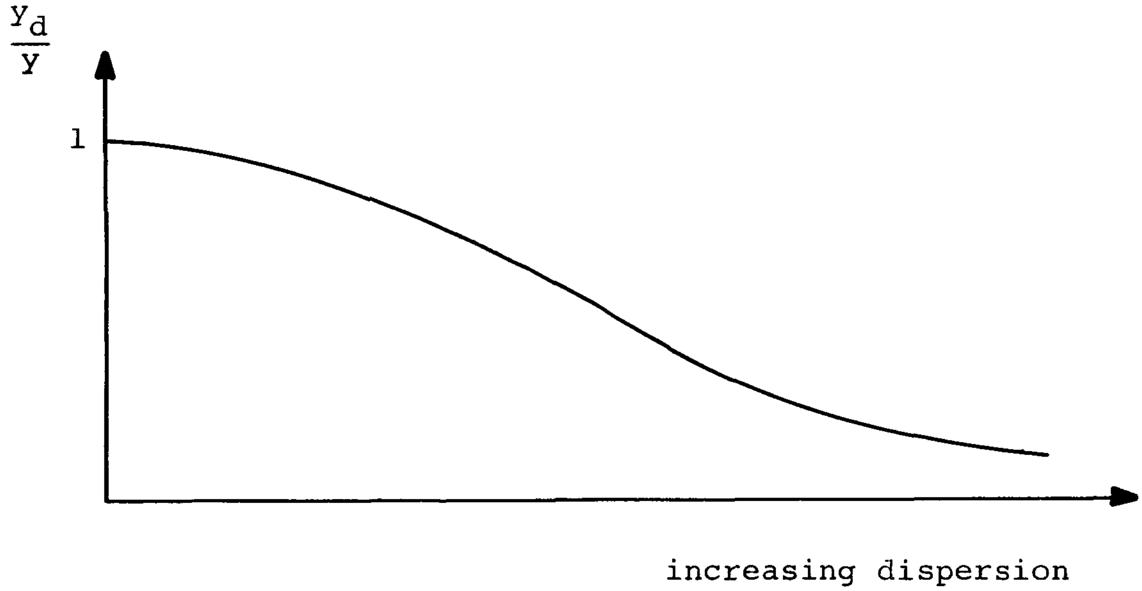


Figure 8

we would then have:

$$\overline{\varepsilon_d^2(\tau)} = \frac{-1}{\frac{2}{N_o}|R_{am}(0)|''} \tag{31}$$

Defining

$$\Phi_{am}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau}|R_{am}(\tau)|d\tau \tag{32}$$

we then have:

$$\overline{\varepsilon_d^2(\tau)} = \frac{\pi}{(\frac{1}{N_o})\int_{-\infty}^{\infty}\omega^2\Phi_{am}(\omega)d\omega} \tag{33}$$

For small signal-to-noise ratio the mean square estimation error would be

$$\overline{\varepsilon_d^2(\tau)} = \frac{-1}{\frac{2}{N_o^2}R_{am}(0)|R_{am}(0)|''}$$

or,

$$\overline{\varepsilon_d^2(\tau)} = \frac{\pi}{(\frac{R_{am}(0)}{N_o^2})\int_{-\infty}^{\infty}\omega^2\Phi_{am}(\omega)d\omega} \tag{34}$$

This expression for $\overline{\varepsilon_d^2(\tau)}$ for small SNR is not very meaningful, however, since it neglects catastrophic error, and, as mentioned earlier, sequential detection will be used to raise the signal energy anyway.

In summary, it should be evident at this point that performance measures for the detection and estimation of the received signal can be evaluated if we know the magnitude of the cross-correlation function $R_{am}(\tau)$. In particular, detection performance is determined by the peak value of this function, and the mean square estimation error can be calculated from the Fourier transform of $|R_{am}(\tau)|$ as in equation 33. The next step, then, is to discuss the calculation of this cross-correlation function. This will be the topic of the next chapter.

III. DISPERSION AND THE CROSS CORRELATION FUNCTION $R_{am}(\tau)$

So far in our discussion of the reception of corrupted signals we have not explicitly stated anything about the nature of the corruption except that it be energy - conserving. We saw that whatever form this corruption took, all that mattered was the resulting cross-correlation function $R_{am}(\tau)$. Our goal now is to find an expression for this $R_{am}(\tau)$ when the particular form of the corruption is dispersion.

Ginzburg (Ref. 7, pps. 403-423) has presented a way of conveniently representing a dispersed signal by employing a Taylor expansion of phase. This approach will be described and used to get an expression for a(t) in terms of m(t). In doing this we will find a parameter that will serve as a good measure for the amount of dispersion. Next we will examine a specific type of dispersion in terms of the medium that causes it - namely, the homogeneous, isotropic, collisionless plasma. It will then be possible to test the assumptions inherent in Ginzburg's approach in the light of the Sunblazer experiment. Finally, we will perform the correlation and obtain an expression for $R_{am}(\tau)$.

A. Representation of the Dispersed Signal.

This section follows Ginzburg's approach of using a Taylor expansion of phase to obtain a convenient expression for the dispersed signal. In this development, $E_o(t)$ is the modulated version of m(t) and E(t) is the modulated a(t).

We can represent a plane wave incident on the dispersive medium by:

$$E_o(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{j\omega t} \, d\omega \tag{35}$$

where,

$$g(\omega) = \int_{-\infty}^{\infty} E_o(t) e^{-j\omega t} \, dt \tag{36}$$

After this signal has propagated a distance z through the dispersive medium, we have:

$$E(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\omega) g(\omega) e^{j[\omega t - \phi(\omega)]} \, d\omega \tag{37}$$

where $R(\omega)$ is an attenuation factor, and $\phi(\omega) = \frac{\omega}{c} n(\omega) z$ is the phase of the signal.

Now if we assume that the signal is quasi-monochromatic, i.e. $\Delta\omega << \omega_o$, we can write:

$$E_o(t) = m(t) e^{j\omega_o t} \tag{38}$$

Now substituting (38) into (36) and changing t to η, we have:

$$g(\omega) = \int_{-\infty}^{\infty} m(\eta) e^{j(\omega_o - \omega)\eta} d\eta \tag{39}$$

Now we substitute this into (37), and we make the assumption that $R(\omega)$ is approximately constant around ω_o and for simplicity set it equal to one:

$$E(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(\eta) e^{j[\omega t + (\omega_o - \omega)\eta - \phi(\omega)]} d\eta d\omega \tag{40}$$

Now we would like to expand $\phi(\omega)$ in a Taylor series about ω_o. We must recognize, however, that our signal, a slowly-varying time function modulating a cosine wave, is double side-band (figure 9).

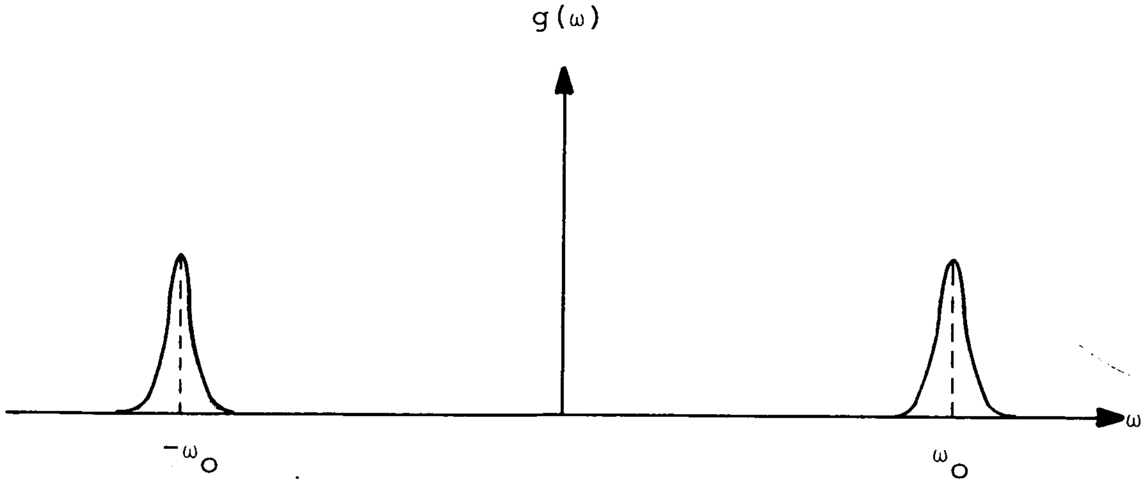


Figure 9

Hence in taking our integral with respect to ω from $-\infty$ to $+\infty$, we must use a different Taylor series representation of $\phi(\omega)$ for $\omega < 0$; i.e. for $\omega < 0$ we must expand $\phi(\omega)$ about $-\omega_o$. In order to avoid this complication we will ignore the left side of the spectrum and integrate with respect to ω from 0 to ∞. In the end, then, we would take the magnitude of our answer instead of the real part. This results in the loss of phase information, but the received signal is phase incoherent anyway. We write then,

$$E(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int^{\infty} m(\eta) e^{j[\omega t + (\omega_o - \omega)\eta - \phi(\omega)]} d\Omega d\eta \tag{41}$$

Now expand $\phi(\omega)$ in a Taylor series about ω_o to three terms:

$$\phi(\omega) \simeq \phi(\omega_o) + \phi'(\omega_o)\Omega + \frac{1}{2}\phi''(\omega_o)\Omega^2 \tag{42}$$

where $\Omega = \omega - \omega_o$. We will see that $\phi(\omega_o)$ will be a measure of the phase delay, $\phi'(\omega_o)$ a measure of the time delay, and $\phi''(\omega_o)$ a measure of the dispersion. Substituting [42] into [41] and making the appropriate change of limits:

$$E(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\omega_o}^{\infty} m(\eta) e^{j[\omega t + (\omega_o - \omega)\eta - \phi(\omega_o) - \phi'(\omega_o)\Omega - \frac{1}{2}\phi''(\omega_o)\Omega^2]} d\Omega d\eta$$

$$-\frac{1}{2}\phi''(\omega_o)\Omega^2] d\Omega d\eta$$

$$= e^{j[\omega_o t-\phi(\omega_o)]}\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\omega_o}^{\infty} m(\eta)\, e^{j[\Omega t-\Omega\eta-\Omega\phi'(\omega_o)-\frac{1}{2}\Omega^2\phi''(\omega_o)]}\ d\Omega\, d\eta \tag{43}$$

We now make the substitution

$$\phi''(\omega_o)\left(\Omega + \frac{\eta-t+\phi'(\omega_o)}{\phi''(\omega_o)}\right)^2 = \pi\xi^2$$

then

$$d\Omega = \sqrt{\frac{\pi}{\phi''(\omega_o)}}\ d\xi, \text{ and}$$

$$\phi''(\omega_o)\Omega^2 + 2\Omega(\eta-t+\phi'(\omega_o)) + \frac{(\eta-t+\phi'(\omega_o))^2}{\phi''(\omega_o)} = \pi\xi^2$$

So,

$$E(t)=e^{j[\omega_o t-\phi(\omega_o)]}\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\frac{\pi}{\phi''(\omega_o)}}\ m(\eta)\, e^{j\frac{(\eta-t+\phi')^2}{2\phi''}}\, d\eta \int_{-\sqrt{\frac{\phi''}{\pi}}\omega_o+\frac{\eta-t+\phi''}{\pi\phi''}}^{\infty} e^{-j\frac{\pi}{2}\xi^2} d\xi \tag{44}$$

$$E(t)=e^{j[\omega_o t-\phi(\omega_o)]}\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\frac{\pi}{\phi''}}\ m(\eta)\, e^{j\frac{(\eta-t+\phi')^2}{2\phi''}}\cdot\left\{\frac{1-j}{2} + F^*\left(\sqrt{\frac{\phi''}{\pi}}\ \omega_o - \frac{\eta-t+\phi''}{\sqrt{\pi\phi''}}\right)\right\}d\eta \tag{45}$$

Here F is the Fresnel integral:

$$F(u) = C(u)+j\ S(u) = \int_0^u e^{j\ \frac{\pi}{2}u^2}\ du$$

$$F^*(u) = C(u)-j\ S(u) = \int_0^u e^{-j\frac{\pi}{2}u^2} du$$

Now making the substitution

$$\frac{(\eta - t + \phi'(\omega_o))^2}{2\phi''(\omega_o)} = \frac{\pi u^2}{2}$$

we have $d\eta = \sqrt{\pi\phi''(\omega_o)}\,du$, and

$$E(t) = e^{j[\omega_o t - \phi(\omega_o)]}\ \frac{1}{2} \int_{-\infty}^{\infty} m(t - \phi'(\omega_o) + \sqrt{\pi\phi''(\omega_o)}\ u)\, e^{j\frac{\pi}{2}u^2} \cdot$$

$$\left\{ \frac{1-j}{2} + F^*\left(\sqrt{\frac{\phi''(\omega_o)}{\pi}}\ \omega_o - u \right) \right\} du \qquad (46)$$

At this point it is worthwhile summarizing the assumptions we made in arriving at [46]:

1. Linearity of the field equations.
2. M(t) is quasi-monochromatic, i.e. $\Delta\omega << \omega_o$.
3. $R(\omega)$ is constant around ω_o.
4. The Taylor expansion of $\phi\ (\omega)$ exists.
5. Higher order terms in this expansion (e.g. $\phi'''(\omega_o)$) can be neglected.

We can simplify (46) significantly if we make the additional assumption that $\sqrt{\frac{\phi''(\omega_o)}{\pi}}\,\omega_o$ is large; i.e. the <u>assumption of large dispersion</u>. We have, then,

$$E(t) = \frac{1-j}{2}\ e^{j[\omega_o t - \phi(\omega_o)]} \int_{-\infty}^{\infty} m(t - \phi'(\omega_o) + \sqrt{\pi\phi''(\omega_o)}\ u)\, e^{j\frac{\pi}{2}u}\ du \qquad (47)$$

which is valid only when:

6. $\sqrt{\frac{\phi''(\omega_o)}{\pi}}\ \omega_o >> 1$; i.e. large dispersion.

Equation (47) is our primary result. We can check it by letting $\phi''(\omega_o) \to 0$ (no dispersion) and recognizing that

$$\int_{-\infty}^{\infty} e^{j\frac{\pi}{2}u^2} du = 1+j$$

Then we get, as expected,

$$E(t) = m(t-\phi'(\omega_o))e^{j[\omega_o t-\phi(\omega_o)]}$$

Here we can recognize that $\phi(\omega_o)$ represents the phase delay and $\phi'(\omega_o)$ the time delay.

At this point I would like to introduce the "dispersion time" as a measure of the amount of dispersion; namely

$$\tau_o = \sqrt{\pi\phi''(\omega_o)}$$

We will see later that this parameter is useful in that it is related to the elementary pulse width of the signal.

B. Assumptions and Approximations - the Coronal Plasma.

If the six assumptions listed previously prove to be valid, then we can use equation (47) to obtain the desired cross-correlation function. In this section we will examine the homogeneous, isotropic, collisionless plasma (a good first order approximation to the extended solar corona) as a dispersive medium. Then we will test and verify our assumptions for the particular case of Sunblazer.

We begin by deriving the dispersion equation for the medium. Writing the first two Maxwell equations:

$$\nabla x \underline{H} = \underline{J} + \frac{\partial \varepsilon_0 \underline{E}}{\partial t} \tag{48}$$

$$\nabla x \underline{E} = -\frac{\partial \mu_0 \underline{H}}{\partial t} \tag{49}$$

For the collisonless plasma we can express the current $\underline{J}$ as:

$$\underline{J} = ne\underline{v} \tag{50}$$

where $\underline{v}$ is the average velocity of the electrons, the equation of motion is:

$$m\frac{d\underline{v}}{dt} = e\underline{E} \tag{51}$$

then,

$$\frac{d\underline{J}}{dt} = ne\,\frac{d\underline{v}}{dt} = \frac{ne^2\underline{E}}{m}$$

$$\frac{\partial}{\partial t}\,(\nabla x \underline{H}) = \frac{\partial \underline{J}}{\partial t} + \frac{\partial^2 \varepsilon_0 \underline{E}}{\partial t^2}$$

$$\nabla x\,\frac{\partial \underline{H}}{\partial t} = \frac{ne^2\,\underline{E}}{m} + \frac{\partial^2 \varepsilon_0 \underline{E}}{\partial t^2}$$

$$\nabla x \frac{1}{\mu_0}\,(\nabla x \underline{E}) = \frac{ne^2\underline{E}}{m} + \frac{\partial^2 \varepsilon_0 \underline{E}}{\partial t^2}$$

If we consider a plane wave solution of the form

$$\underline{E} = \underline{i}_x\,E_0 e^{j(\omega t - kz)} \tag{52}$$

we have for the wave equation

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{ne^2 \mu_o E_x}{m} + \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} \tag{53}$$

Substituting (52) into (53) gives us the dispersion equation:

$$k^2(\omega) = \frac{\omega^2}{c^2} - \frac{ne^2\mu_o}{m} = \frac{\omega^2 - \frac{ne^2\mu_o c^2}{m}}{c^2}$$

Defining the "plasma frequency",

$$\frac{ne^2\mu_o c^2}{m} = \omega_p{}^2 = \frac{ne^2}{m\varepsilon_o} \tag{54}$$

The dispersion equation becomes:

$$k(\omega) = \frac{1}{c}\sqrt{\omega^2 - \omega_p{}^2} \tag{55}$$

Equation 55, plotted in figure 10, completely describes the plasma in terms of a dispersive medium. When $\omega < \omega_p$, $k(\omega)$

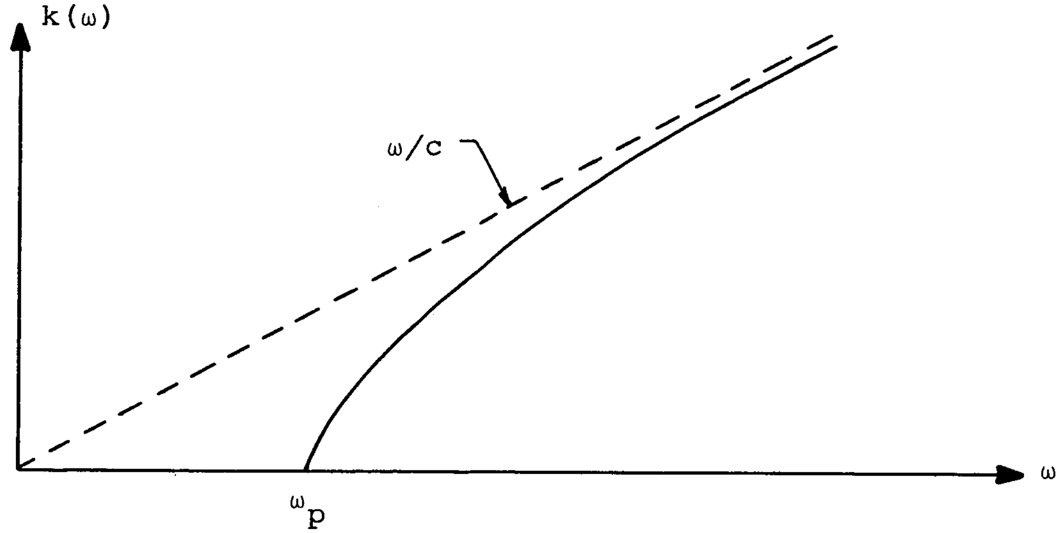


Figure 10 - Dispersion Equation for Collisionless Plasma

becomes imaginary and the wave attenuates (non-uniform plane wave). When $\omega > \omega_p$ the wave propagates without attenuation. The closer ω is to ω_p the greater is the second derivative of $k(\omega)$ and thus the greater is the amount of dispersion. As $k(\omega)$ gets very large, $k(\omega) \rightarrow w/c$, and the amount of dispersion approaches zero.

To get some values for ω_p for the solar corona, we need some numbers for the electron density. If n is expressed as the number of electrons per cc, we find that

$$\omega_p \simeq 5.6 \cdot 10^4 \cdot n^{1/2} \text{ radians/sec.}$$

or

$$f_p \simeq 9 \cdot 10^3 \cdot n^{1/2} \text{ cps.}$$

To get some numerical vlaues for n we use the Allen-Baumbach model for the solar corona (Ref. 9). This assumes a spherically symmetrical corona, so that n is only a function of radial distance. Specifically, for $r < 1.5\ R_o$:

$$n(r) \simeq 10^8 \left[\frac{1.55}{r^6} + \frac{2.99}{r^{16}}\right] \text{ electrons/cc} \tag{56}$$

and for $1.5R_o < r < 50\ R_o$:

$$n(r) \simeq \frac{10^8}{r^6} + \frac{10^6}{r^2} \text{ electrons/cc} \tag{57}$$

where r is in solar radii. Also, direct measurements have shown that in the vicinity of the earth ($r \simeq 215R_o$), $n \simeq 10$ electrons/cc. We find, then, that at $r = 3R_o$, $f_p \simeq 4.5$Mcps; and at $r=1.5R_o$, $f_p \simeq 30$ Mcps. Of course in the event of such occurrences as solar flares and prominences, the electron density can deviate enormously from these estimated values.

Clearly $k(\omega)$ lends itself to a Taylor series expansion. What we would like to show is that we can neglect the higher order terms in this expansion, i.e. that we can write equation 42 to good approximation. To do this we simply expand $\phi(\omega) = k(\omega)z$ to four terms and show that

$$\left|\frac{\text{4th term}}{\text{3rd term}}\right| << 1$$

i.e. $$\left|\frac{k'''(\omega_o)}{3k''(\omega_o)}\right| << 1$$

I will not go through this calculation here, but only indicate the results.

For $r=1.5\ R_o$, $\begin{cases} \omega_o = 500 \times 10^6 \\ \omega_p = 200 \times 10^6 \\ \Delta\omega = 5 \times 10^6 \end{cases}$, we find that:

$$\left|\frac{k'''(\omega_o)\Delta\omega}{3k''(\omega_o)}\right| = .006<<1$$

Also, as ω_p becomes smaller, this ratio asymtotically approaches a small value:

$$\lim_{\omega_p \to 0} \left|\frac{k'''(\omega_o)\Delta\omega}{3k''(\omega_o)}\right| = \frac{\Delta\omega}{2\omega_o} <<1.$$

Then, to well within 1%, we can ignore the fourth term in the Taylor expansion.

Next we would like to look at a second order model for the coronal plasma and examine the attenuation due to collisions. It is shown in the appendix that the coefficient of attenuation due to collisions is given by:

$$k = \frac{\nu \left(\frac{\omega_p}{\omega}\right)^2}{2c\sqrt{1-\left(\frac{\omega_p}{\omega}\right)}} \tag{58}$$

Here ν is the effective electron-ion collision frequency, given by (Ref. 9, pps. 35-38):

$$\nu \simeq 42 \cdot 10^6 \cdot nT^{-3/2}$$

in the corona. The total attenuation along a ray path is then

$$A(\rho) = 2\int_\rho^\infty \kappa ds$$

At a frequency of 100 Mcps, and $T = 10^6$ °K, $A(\rho)$ is 1 db at $\rho = 1.5R_o$, is .02 db at $2R_o$, and zero for $\rho \geq 4\ R_o$. Hence this attenuation does not play a large role. Nevertheless, if we look at the variation in this attenuation over $\Delta\omega = 5\text{x}10^6$, we find that

$$\frac{\Delta R(\omega)}{\Delta\omega} \simeq 1\%$$

Our assumption is valid, then, that $R(\omega)$ is approximately constant over $\Delta\omega$. This of course is really a result of the fact that $\Delta\omega << \omega$.

As far as the linearity of the field equations is concerned, there are certainly non-linear effects that can occur in the propagation of a signal through a dilute plasma (e.g. Luxemburg effect). I will only say here that we need not worry about such effects because of the low signal power, and the fact that magnetic fields present are weak.

Finally, we would like to investigate the approximation of large dispersion. We want to show that

$$\sqrt{\frac{\phi''(\omega_o)}{\pi}}\,\omega_o = \sqrt{\frac{k''(\omega_o)z}{\pi}}\,\omega_o >> 1$$

It should be clear from the expression above that the approximation of large dispersion is really an approximation of long path length.

Rather than integrate $\sqrt{k''(\omega_o)z}$ over a changing ω_p, I have made the crude but conservative approximation that ω_p is constant at its value at $r = \rho$ for path length $z = 5R_o$ and zero elsewhere. This gives:

$$\sqrt{\frac{\phi''(\omega_o)}{\pi}}\,\omega_o \simeq \omega_p \cdot 10^{-4}$$

This becomes ≤ 10 only for $\omega_p \leq 10^5$, or $n \leq 3$ electrons/cc. And this is extremely conservative - for large ρ (small dispersion), z should be taken much larger than $5R_o$. In any case, the smallest n that we will encounter will be 10 electrons/cc.

Hence the approximation of large dispersion will always be valid for any point in the Sunblazer trajectory.

What we have done in this section is to derive the dispersion equation for the coronal plasma, and show that equation 47 can be used to meaningfully represent the dispersed Sunblazer signal. Now we can use this to get the cross-correlation function, $R_{am}(\tau)$.

C. <u>The Cross-Correlation Function</u>.

Let us rewrite equation 47 adding on the random phase angle θ:

$$E(t) = \frac{1-j}{2} e^{j[\omega_o t - \phi(\omega_o) + \theta]} \int_{-\infty}^{\infty} m(t - \phi'(\omega_o) + \tau_o u) e^{j\frac{\pi}{2}u^2} du \tag{59}$$

Now define the receiver time, $t' = t - \phi'(\omega_o)$; i.e. t' is the time measured from the signal's arrival at the observation point, disregarding dispersion. Then,

$$E(t') = \frac{1-j}{2} e^{j[\omega_o t' + \omega_o \phi'(\omega_o) - \phi(\omega_o) + \theta]} \int_{-\infty}^{\infty} m(t' + \tau_o u) e^{j\frac{\pi}{2}u^2} du \tag{60}$$

Or, after demodulation,

$$a(t') = \frac{1-j}{2} e^{j[\omega_o \phi'(\omega_o - \phi(\omega_o + \theta]} \int_{-\infty}^{\infty} m(t' + \tau_o u) e^{j\frac{\pi}{2}u^2} du \tag{61}$$

We define the cross-correlation function:

$$R_{am}(\tau) = \int_{-\infty}^{\infty} a^*(t'-\tau)m(t')dt' \tag{62}$$

Then,

$$R_{am}(\tau) = \frac{1+j}{2}e^{-j[\omega_o\phi'(\omega_o)-\phi(\omega_o)+\theta]}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(t')m(t'-\tau+\tau_o u)e^{-j\frac{\pi}{2}u^2}dt'du \tag{63}$$

$$R_{am}(\tau) = \frac{1+j}{2}e^{-j[\omega_o\phi'(\omega_o)-\phi(\omega_o)+\theta]}\int_{-\infty}^{\infty} R_{mm}(\tau-\tau_o u)e^{-j\frac{\pi}{2}u^2}du \tag{64}$$

Letting $\theta' = \omega_o\phi'(\omega_o)-\phi(\omega_o) + \theta$,

$$R_{am}(\tau) = \frac{1+j}{2}e^{-j\theta'}\int_{-\infty}^{\infty} R_{mm}(\tau-\tau_o u)e^{-j\frac{\pi}{2}u^2}du \tag{65}$$

We want

$$y(\tau) = |R_{am}(\tau)| = \left(R_{am}^*(\tau)\cdot R_{am}(\tau)\right)^{1/2}$$

$$= \left\{(\mathrm{Re}[R_{am}(\tau)])^2 + (\mathrm{Im}[R_{am}(\tau)])^2\right\}^{1/2} \tag{66}$$

Then,

$$y(\tau) = \frac{1}{\sqrt{2}}\left\{\left[\int_{-\infty}^{\infty} R_{mm}(\tau-\tau_o u)\cos\frac{\pi}{2}u^2du\right]^2 + \left[\int_{-\infty}^{\infty} R_{mm}(\tau-\tau_o u)\sin\frac{\pi}{2}u^2du\right]^2\right\}^{1/2} \tag{67}$$

Equation 67 is our desired result. It gives the magnitude of the cross-correlation function in terms of the auto-correlation function of m(t) and the dispersion τ_o. This expression has been evaluated over a range of τ_o for three different m(t) - a rectangular pulse, a 3-bit Barker code, and an 11-bit Barker code. Plots of these $y(\tau)$ appear in the next chapter. Also, an evaluation of τ_o for different points in the proposed Sunblazer trajectory has been done and appears as an appendix.

I would now like to make one observation about $y(\tau)$ that applies whenever m(t) is a binary code with baud length (elementary pulse width) T. If this is the case, the degree of corruption caused by dispersion depends only on the parameter $\beta = \tau_o/T$ called the dispersion ratio. For example, consider the ratio $\alpha = y_d/y$ from equation 30.

$$\alpha^2 = \frac{\frac{1}{2}\left[\int_{-\infty}^{\infty} R_{mm}(\tau_o u)\cos\frac{\pi}{2}u^2 du\right]^2 + \frac{1}{2}\left[\int_{-\infty}^{\infty} R_{mm}(\tau_o u)\sin\frac{\pi}{2}u^2 du\right]^2}{R_{mm}^2(0)}$$

For simplicity in writing we break up the sum and consider just the cosine part.

$$\alpha \sim \int_{-\infty}^{\infty} R_{mm}(\tau_o u)\cos\frac{\pi}{2}u^2\, du \Big/ R_{mm}(0)$$

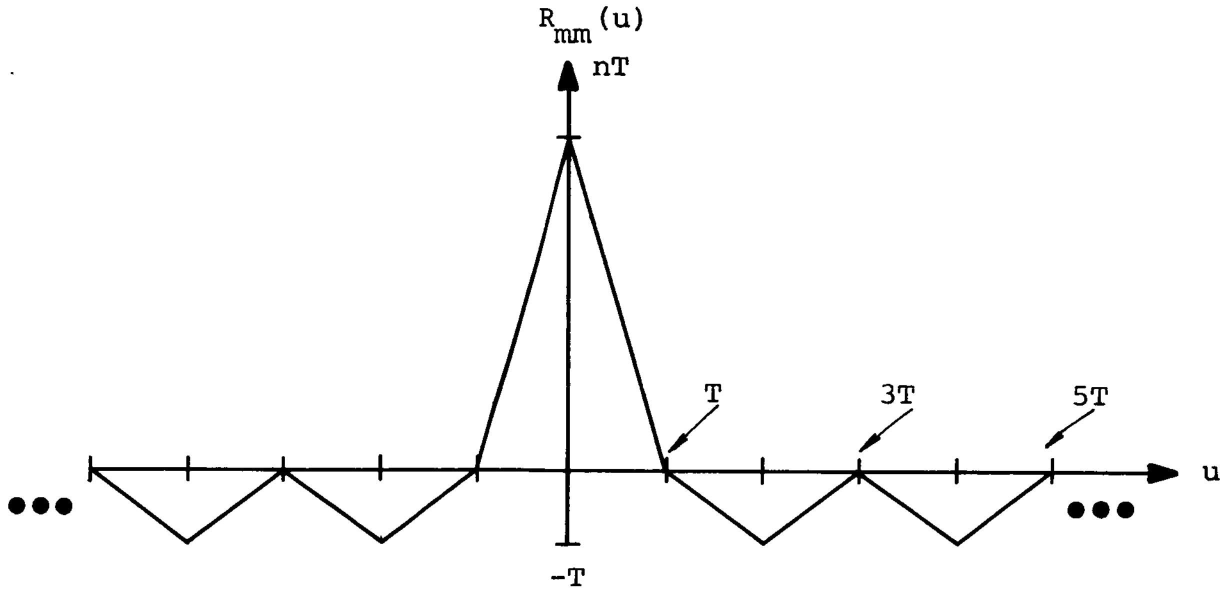


Figure 11a

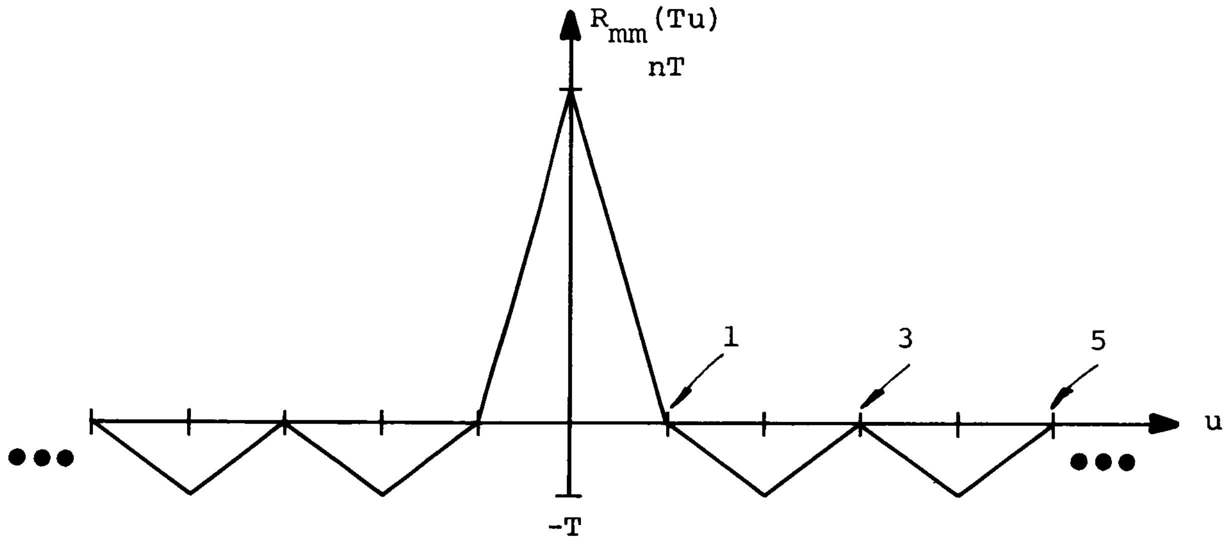


Figure 11b

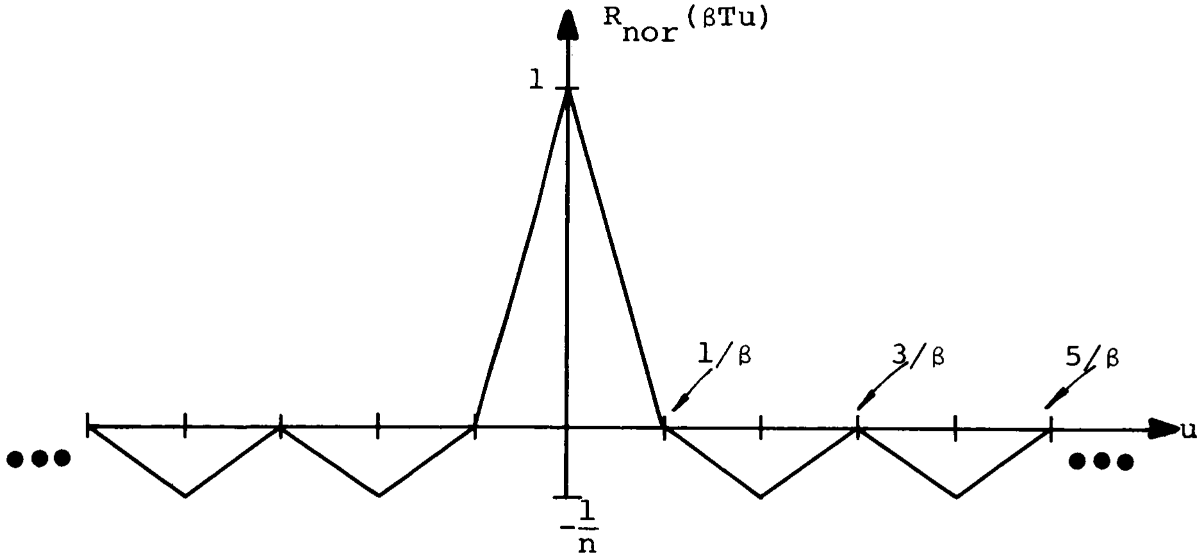


Figure 11c

But $R_{mm}(0) = nT$, where n is the number of bits (figure 11a).

Then
$$\alpha \sim \int_{-\infty}^{\infty} R_{mm}(\tau_o u) \cos\frac{\pi}{2}u^2 du \Big/ nT$$

Let $\tau_o = \beta T$. Then $R_{mm}(\tau_o u) = R_{mm}(\beta Tu)$. $R_{mm}(\tau_o u)$ intersects the u-axis at $1/\beta$, $3/\beta$, etc. Then only the height of $R_{mm}(\tau_o u)$ depends on T, and we can normalize this:

$$R_{nor}(\tau_o u) = \frac{1}{nT} R(\tau_o u) \qquad \text{(figure 11c)}$$

then
$$\alpha \sim \frac{nT \int_{-\infty}^{\infty} R_{nor}(\beta Tu) \cos\frac{\pi}{2}u^2 du}{nT}$$

$$\alpha \sim \int_{-\infty}^{\infty} R_{nor}(\beta Tu) \cos\frac{\pi}{2}u^2 du$$

The term $\cos \frac{\pi}{2} u^2$ is independent of T, τ_o, and β. And we have just seen that $R_{mm}(\beta TU)$ is dependent only on β. Hence α is a function only of β.

Similarily we could show that $y_d(\tau) / y(\tau)$ for any τ is a function only of β. This, of course, is what provides most of the motivation for using τ_o as a measure of dispersion.

IV. RESULTS

This chapter presents the computational results for the numerical calculation of $y(\tau) = |R_{am}(\tau)|$, and its application to an evaluation of the performance measures described earlier. Three cases for m(t) are examined--the rectangular pulse, and the three and eleven-bit Barker codes. In each case the peak value of the cross-correlation is first computed for different values of $\beta = \tau_o/T$. Then the entire cross-correlation function is computed and plotted for different β. Finally, these calculations are used to compute the mean-square estimation error, $\overline{\varepsilon^2(\tau)}$, for large signal-to-noise ratio.

A. Computational Methods

Most of the computations were performed using the MAP (Mathematical Analysis Program) system through Compatible Time Sharing on the IBM 7094 computer. MAP, designed for the solution of mathematical problems, replaces the normal procedures of programming with direct command-response interchange between the user and the computer. For a detailed description of the system, I refer the reader to the MAP manual (Ref. 12).

Besides the simplicity of MAP, one of its big advantages was that it saved a good deal of programming time by making routines for integration, Fourier transform, etc.

readily available. Also, MAP provides a plotting routine in which up to three functions can be plotted on the same set of axes. This made it possible to obtain a plot for the cross-correlation function $y(\tau)$ as soon as it had been calculated.

One of the problems with MAP in its present form is that there are a limited number of things that the system will do. For example, MAP cannot handle functional arrays of more than one dimension (this is to be changed in the near future). For this reason I found it necessary to write my own programs in MAD for the actual calculations of $y(\tau)$. Once these programs had been filed in memory, however, they could be executed within MAP via MAP commands. Another limitation in MAP is that it will only accept up to 1000 data points for the definition of a function. This is insufficient for problems much more complex than an eleven-bit code. It also puts limitations on the parameter β. If β is very small, the sine and cosine terms in equations 67 change extremely rapidly with respect to the auto-correlation function, and a large number of data points are required for a meaningful answer.

Another important point worth mentioning is that these computations consume large amounts of computer time. To calculate $y(\tau)$ with enough data points to allow for a calculation of its Fourier transform, as well as a plot, requires hundreds of integrations. Calculations of a $y(\tau)$ would typically use from four to five minutes of time, depending on the range and number of data points desired. Additional time is consumed in the calculation of $\overline{\varepsilon^2(\tau)}$ and in the plotting routines. Because of this sheer expenditure of computer time (and money), the calculation of $y(\tau)$ and its corresponding $\overline{\varepsilon^2(\tau)}$ was performed for only four or five values of β for each of the three cases. The results, then, at least for the calculation of $\overline{\varepsilon^2(\tau)}$, are limited in terms of detailed accuracy.

I would like to stress at this point that accurate answers, in terms of a number of decimal places, is not the aim of this thesis. The reason for spending time and money on these computations is to give a qualitative idea of how the communication scheme is affected by dispersion. This, I feel, has been accomplished successfully.

B. Detection - Calculation of y(0).

We begin by rewriting equation 67 in a form that is somewhat more amenable to computation. Recognizing that $R_{mm}(X) = R_{mm}(-X)$, and making the transformation $v = \tau_o u-\tau$, we have:

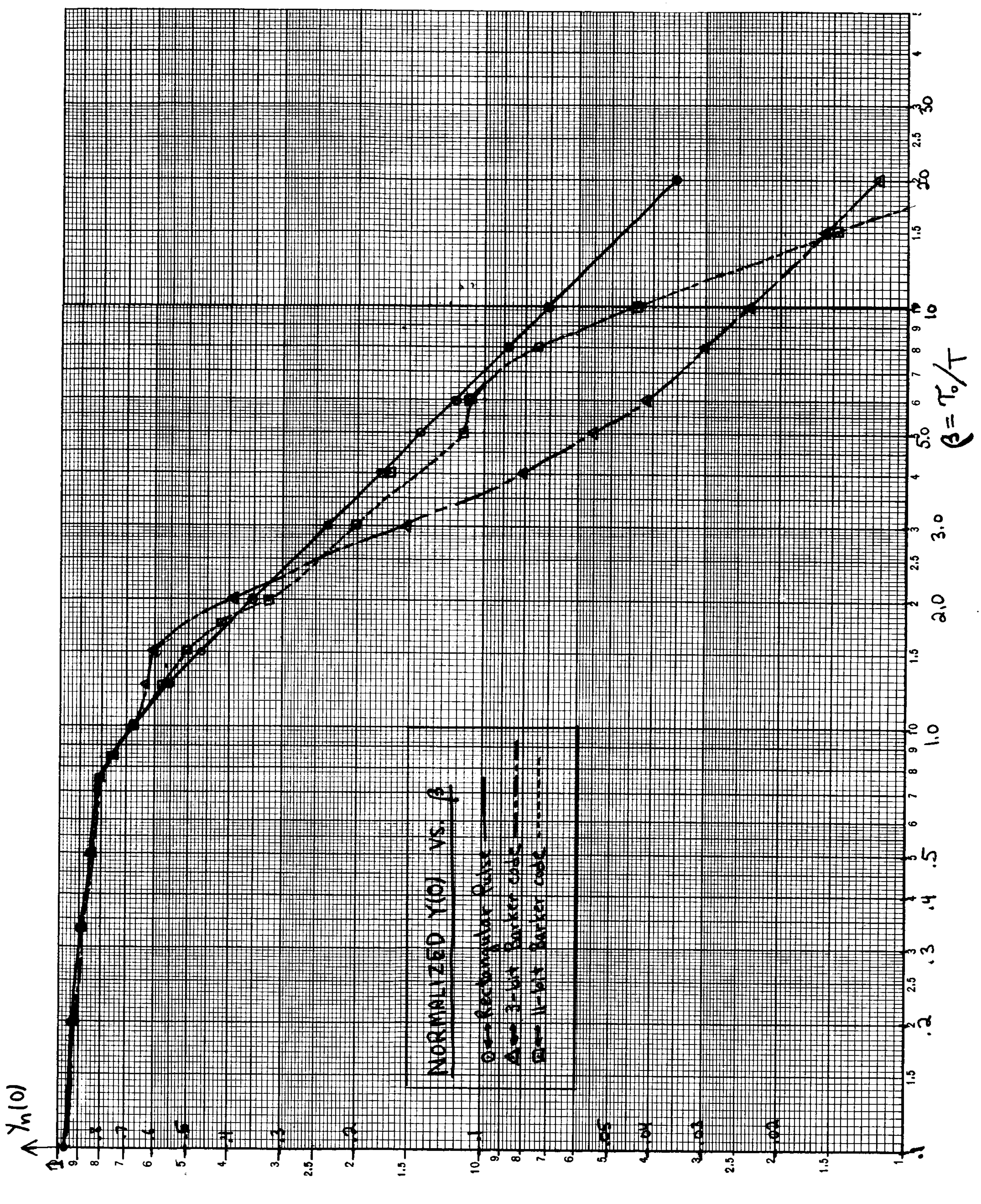
$$y(\tau) = \frac{1}{\sqrt{2}}\left\{\left[\frac{1}{\tau_o}\int_{-\infty}^{\infty} R(v)\cos\frac{\pi}{2}\frac{(v+\tau)^2}{\tau_o^2}dv\right]^2 + \left[\frac{1}{\tau_o}\int_{-\infty}^{\infty} R(v)\sin\frac{\pi}{2}\frac{(v+\tau)^2}{\tau_o^2}dv\right]^2\right\}^{1/2} \tag{68}$$

The auto-correlation function R(v) was computed by giving a MAP command to convolve m(t) and m(-t). A single

MAD program ("CROSO"-see Appendix D) was written to calculate $y(0)$ for a given value of τ_o. To change τ_o, the MAP command "delete τ_o" was given; MAP would then ask for the value of τ_o when it tried to execute the program.

The results of these computations are plotted on the next page for each of the three cases. Here the dispersed $y(0)$ is normalized with respect to the undispersed $y(0)$; what we are plotting in effect then is the parameter α from equation 30. Note that for all of the cases there is a sharp roll-off beginning at β just less than one, or $\tau_o \simeq T$. Here we see another significance for the parameter τ_o. Just as the baud length T is a measure of the signal bandwidth, so is τ_o a measure of the "bandwidth" of the medium when viewed as a filter (Ref. 6, p. 18). Suppose τ_o was fixed, and we tried to transmit a signal with $T = \tau_o/2$; i.e. twice the bandwidth of the medium. The signal would disperse, resulting in its bandwidth becoming smaller--to a rough approximation, no bigger than the bandwidth of the medium. Hence, the dispersive medium limits the bandwidth that our received signal can have, and if the signal energy is constrained, limits the best mean-square estimation error that we can hope to achieve, no matter how we code the signal. This phenomenon of bandwidth contraction by dispersion is well illustrated by the photographs in the next section, in which the Fourier transform of the dispersed $y(\tau)$ is plotted for different β.

Another point about the plots of $y(0)$ worth mentioning is that the curve for the three-bit code falls off much more


NORMALIZED Y(0) VS. β
Rectangular Pulse
3-bit Barker code
11-bit Barker code
$Y_n(0)$
$\beta = \tau_o/T$

rapidly for large dispersion than does the one for the rectangular pulse. The reason for this is that the auto-correlation function for the three-bit code (photograph P8) has large negative side peaks next to the main peak. When the dispersion is large and all the peaks are smeared out, the smeared side peaks add negatively to the smeared center peak, resulting in a still smaller $y(0)$. This, of course, does not occur with the rectangular pulse whose auto-correlation function is simply a single triangle. The effect does occur with the eleven-bit code, but, as is evident from the plot, to a much lesser degree since the side peaks are so much smaller than the main peak.

As far as the accuracy of these results is concerned, I estimate that every point on the curves is accurate to well within 2%. The calculation was simple, and a large number of data points were taken for the integrations.

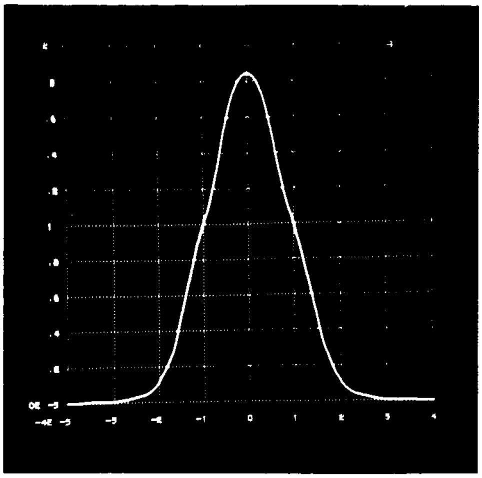
C. Calculation of $y(\tau) = |R_{am}(\tau)|$.

Again, a MAD program was written to evaluate equation 68 for $y(\tau)$, but this time for up to a few hundred values of τ (the number depending on the complexity of $R(v)$ and the range of $y(\tau)$). As before, a MAD program ("CRSCOR"--see Appendix D) was written to calculate $y(\tau)$ for a given value of τ_o. Then both $y(\tau)$ and $P(v) = |R(v)|$ were plotted on the same set of axes for comparison. The Fourier transform of $y(\tau)$ was calculated each time, and for some of the cases was plotted in order to show the effect of bandwidth compression.

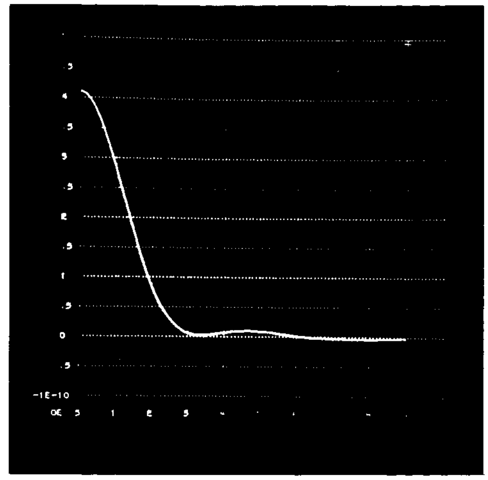
The photographed plots which follow are, I feel, an extremely important part of the results of this thesis. They are probably about the best means possible of giving a qualitative impression of how the signal, or more specifically the output of the correlation receiver, is affected by different amounts of dispersion. As for accuracy, all the plots of $y(\tau)$ are good to within at least 5% at every point. The plots of the Fourier transforms are accurate to within 10%.

There are a few observations that should be made with regard to the plots. First, we can again see that $\tau_o = T$, or $\beta = 1$, marks a kind of critical point in the degradation of $y(\tau)$. For example, in plots P10 and P11 we see that for $\beta = 1/2$, $y(\tau)$ deviates only slightly from $P(v)$, but for $\beta = 1$ the deviation is considerable. Also, we see that once β has reached 5 (P7), estimating the arrival time (assuming we could even detect the signal) with any meaningful resolution becomes virtually impossible.

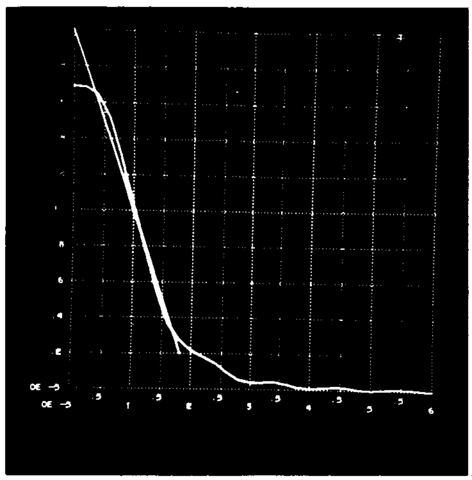
It is also interesting to note that for the 3 and 11 bit codes, when $\beta = 5, y(\tau)$ no longer has its maximum at $\tau = 0$ (plots P14, P24). Instead, the central peak is gone, and larger side peaks appear. One result of this, of course, is that a "catastrophic error" is made in estimating the arrival time. Another result, though, is that the probability of successfully detecting the signal is higher than what we would expect by looking only at $y(0)$. Compare plots P7 and P14. If we base our detection decision on some peak output of the correlation receiver, no matter where that peak occurs, then for the same amount of dispersion and



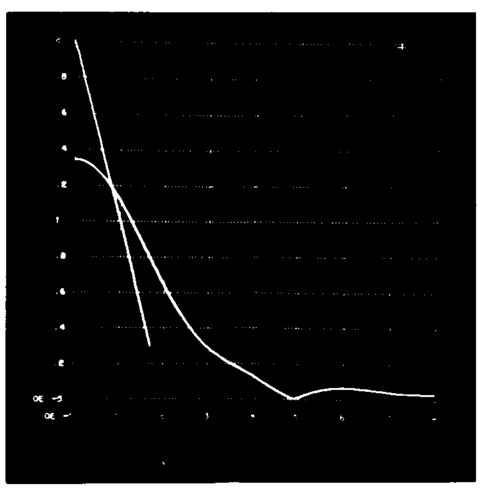
P1 - Rectangular pulse. $y(\tau)$ for $\tau_0 = T/4$.



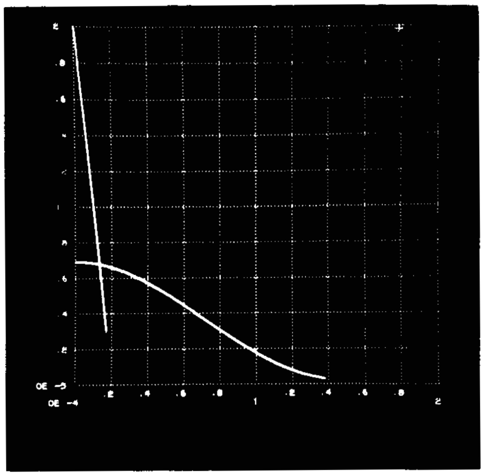
P2 - Corresponding spectrum, $\tau_c = T/4$.



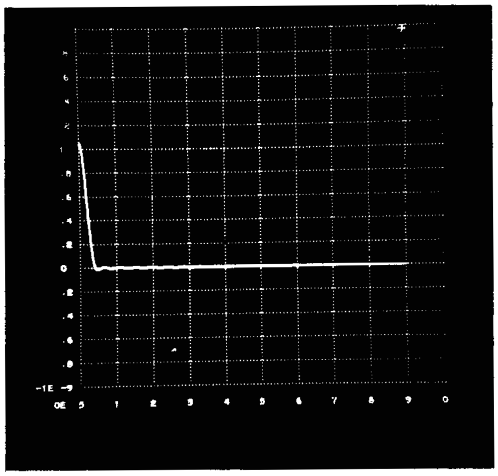
P3 - $y(\tau)$ for $\tau_0 = T/2$, and $P(v)$



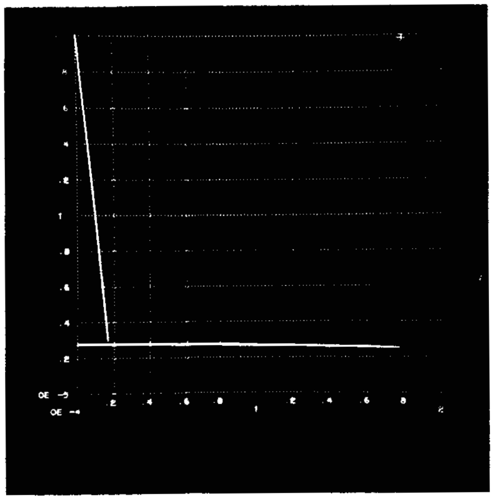
P4 - $y(\tau)$ for $\tau_c = T$, and $P(v)$.



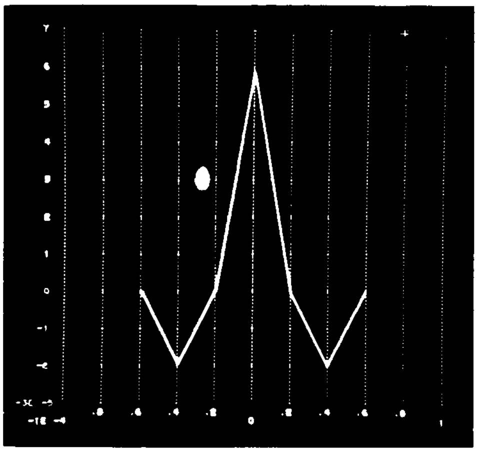
P5 - $y(\tau)$ for $\tau_i = 2T$, and $P(v)$



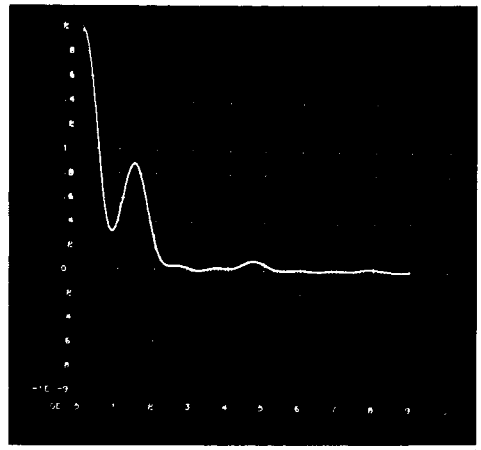
P6 - Corresponding spectrum, $\tau_o = 2T$



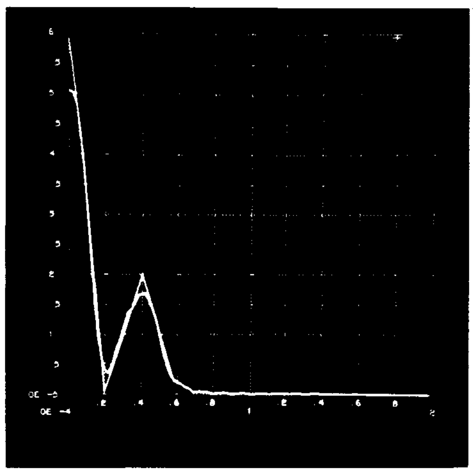
P7 - $y(\tau)$ for $\tau_o = 5T$, and $P(v)$



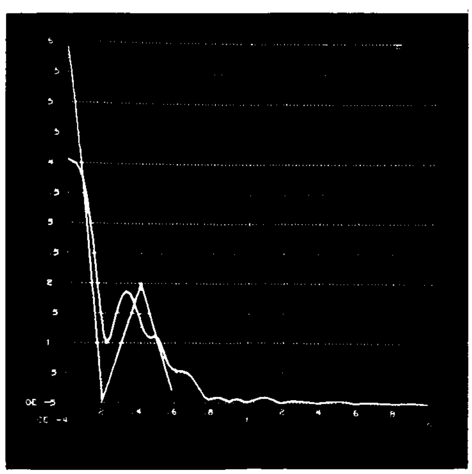
P8 - R(v) for 3-bit Barker code.



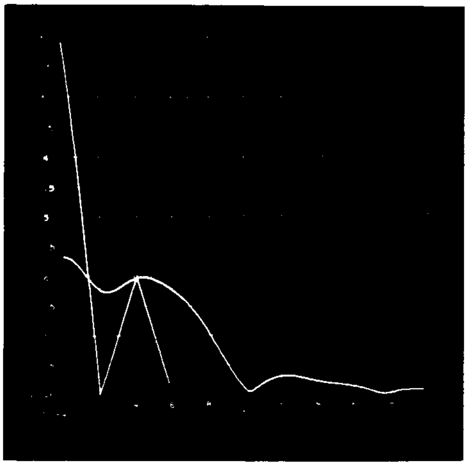
P9 - Corresponding spectrum for P(v) = |R(v)|



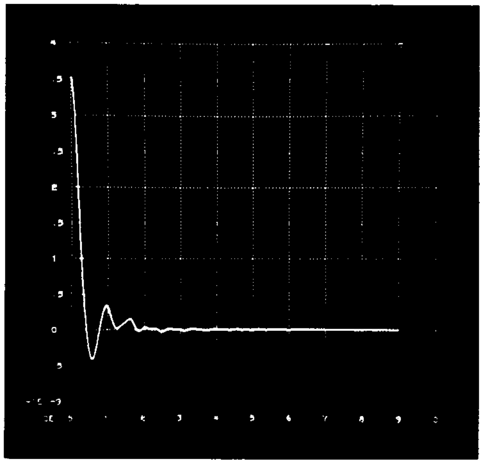
P10 - y(τ) for τ_0 = T/2, and P(v)



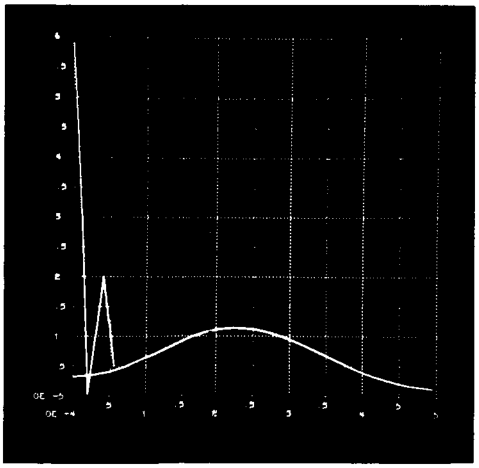
P11 - y(τ) for τ_0 = T, and P(v)



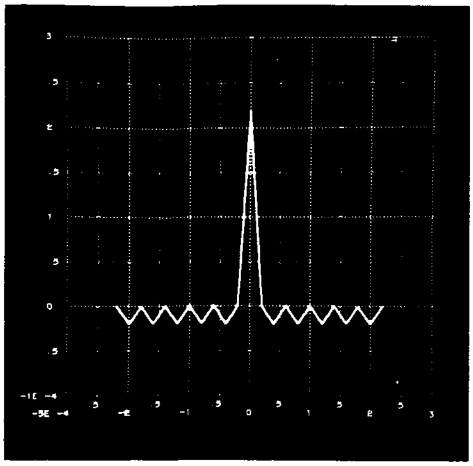
P12 - $y(\tau)$ for $\tau_0 = 2T$, and $P(v)$



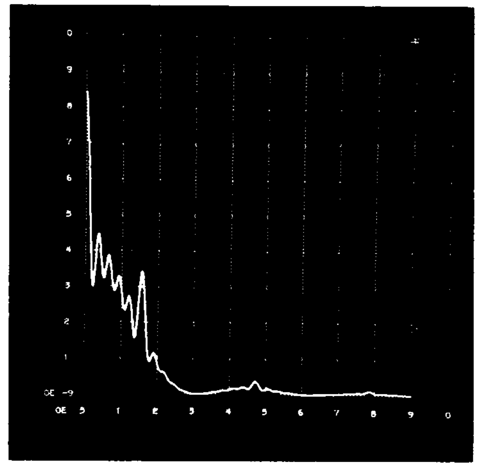
P13 - Corresponding spectrum, $\tau_0 = 2T$



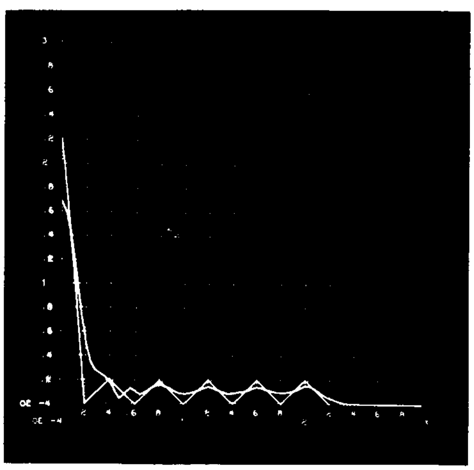
P14 - $y(\tau)$ for $\tau_0 = 5T$, and $P(v)$



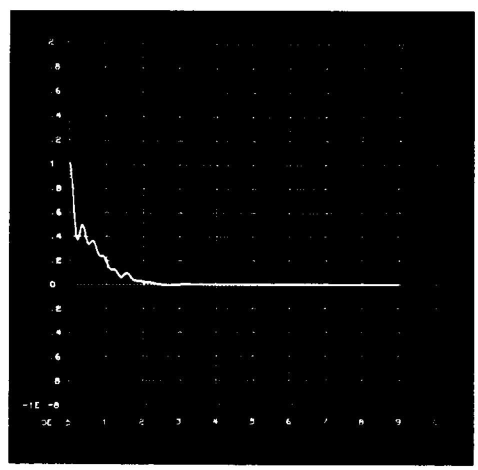
P15 - R(v) for the 11-bit Barker code



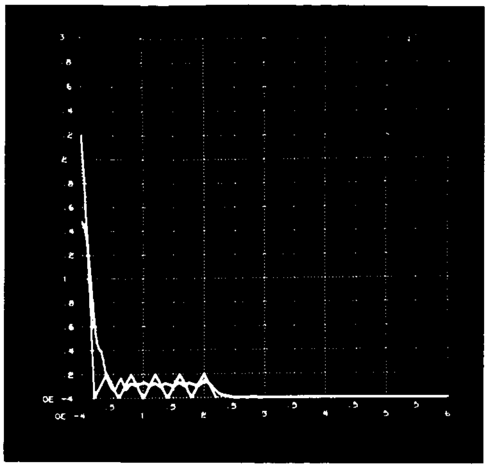
P16 - Corresponding spectrum for $P(v) = |R(v)|$



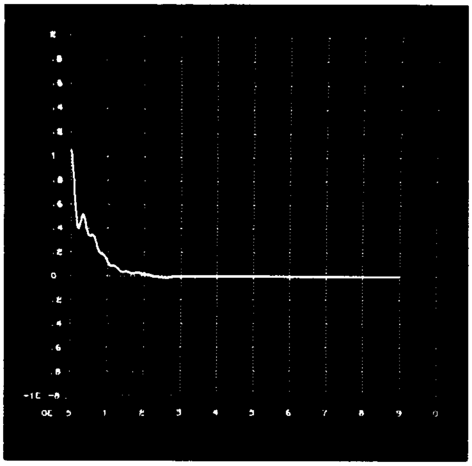
P17 - $y(\tau)$ for $\tau_0 = .85T$, and $P(v)$



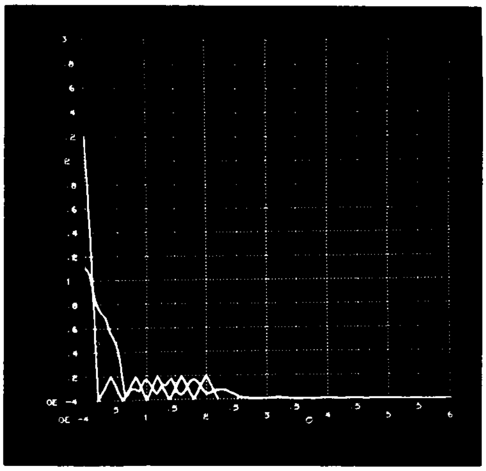
P18 - Corresponding spectrum, $\tau_0 = .85T$



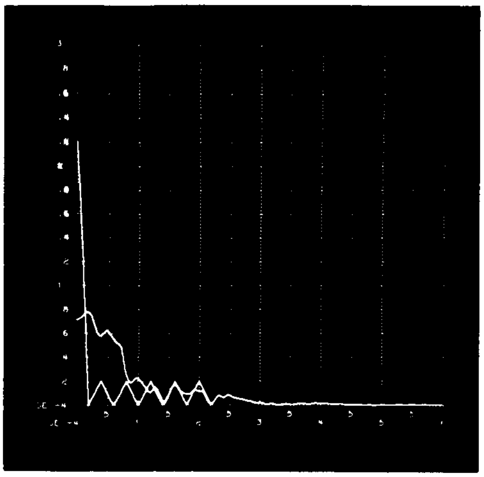
P19 - y(Υ) for Υ_o = T, and P(v)



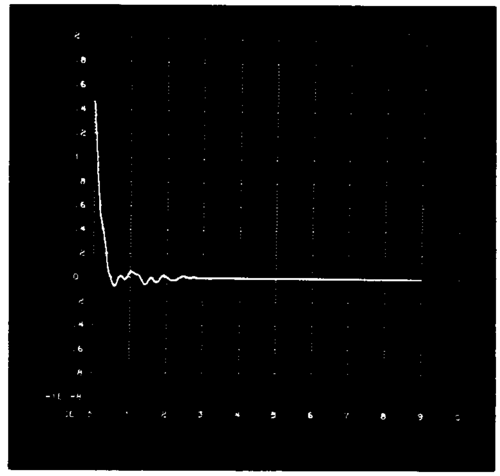
P20 - Corresponding spectrum, Υ_o = T



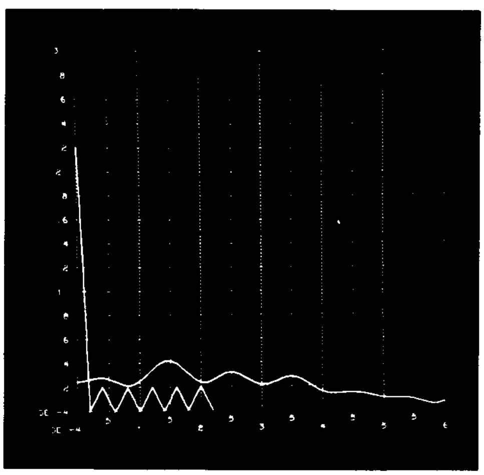
P21 - y(Υ) for Υ_o = 1.5T, and P(v)



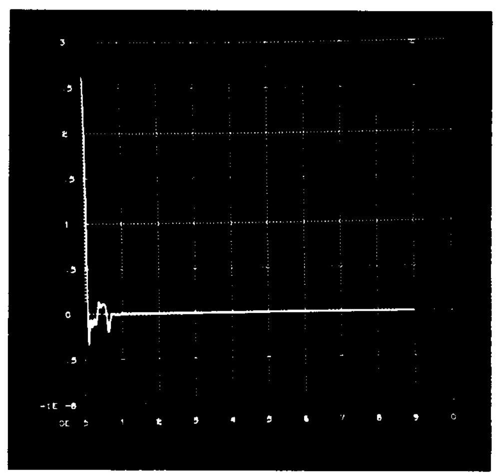
P22 - $y(\tau)$ for $\tau_0 = 2T$, and $P(v)$



P23 - Corresponding spectrum, $\tau_0 = 2T$



P24 - $y(\tau)$ for $\tau_0 = 5T$, and $P(v)$



P25 - Corresponding spectrum, $\tau_0 = 5T$

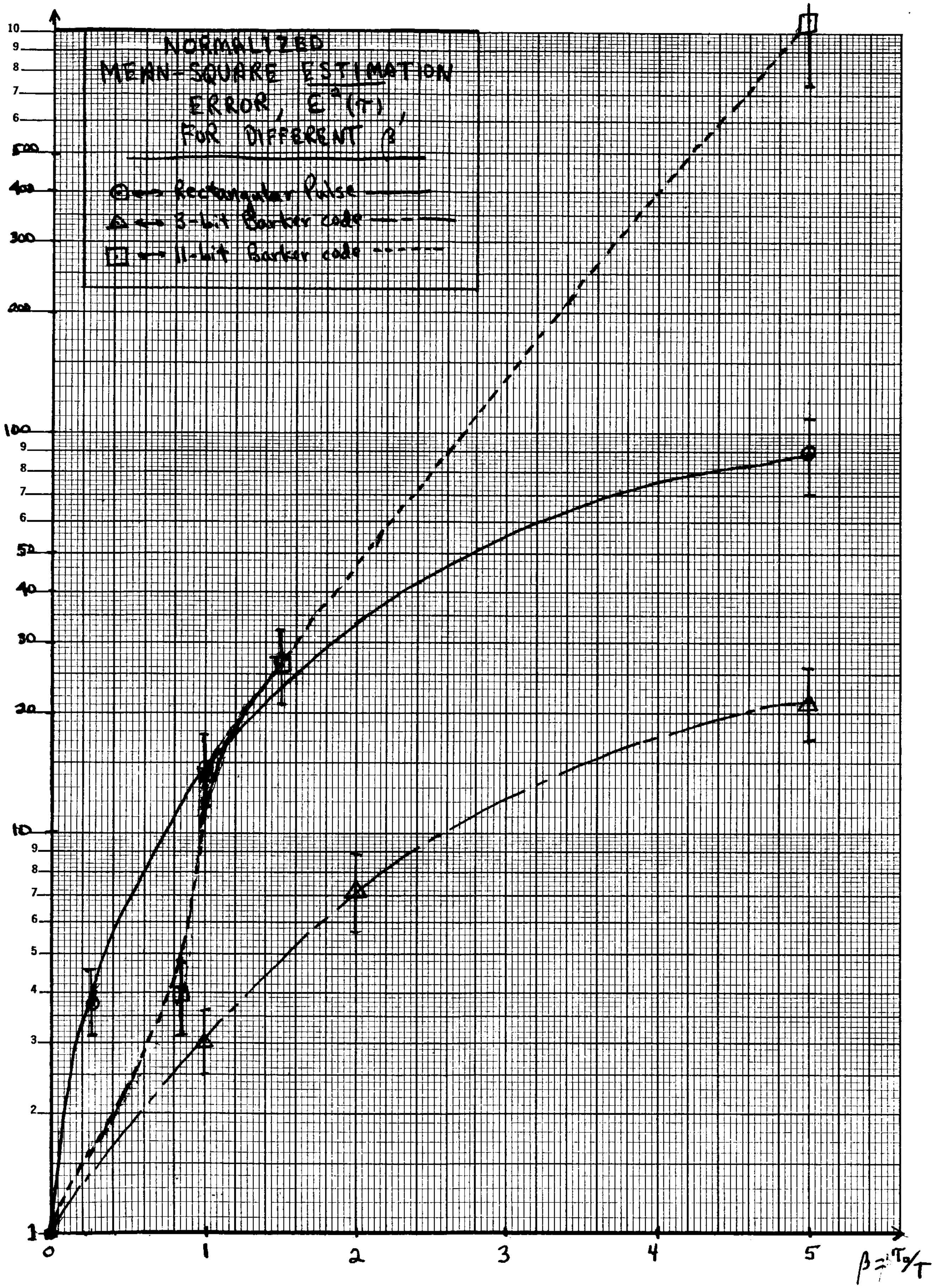
same signal energy it would be easier to detect the 3-bit code than the rectangular pulse.

This argument is certainly very qualitative, and the resulting difference in detection probabilities is small. I brought it out mainly to illustrate that for dispersive channels coding can influence signal detection, and parameter estimation may depend not only on the signal energy and bandwidth, but on the exact form of the code. I will not pursue this any longer at this time, but I would point out that further work along these lines is in order.

D. Estimation - Calculation of $\overline{\varepsilon^2(\tau)}$.

Once $y(\tau)$ had been calculated, the mean-square estimation error could then be computed using only MAP commands. The Fourier transform of $y(\tau)$ was computed (call it $c(w)$, a new function $x(w) = w^2 \cdot c(w)$ was created, and this function was then integrated. A normalized $\overline{\varepsilon^2(\tau)}$ could then be obtained by dividing the integral resulting from the undispersed case by this integral. The results are plotted on the next page.

Again, these calculations are designed only to give a rough idea of how a particular kind of estimation error is affected by dispersion. In this sense their value is limited and they shouldn't be taken too seriously. For example, note that at $\beta = 5$ the error for the 3-bit code is considerably smaller than that for the rectangular pulse. We would expect this from comparing plots P7 and P14 in the previous section--$y(\tau)$ for the 3-bit code has a peak on each side, while $y(\tau)$


NORMALIZED MEAN-SQUARE ESTIMATION ERROR, $\overline{\epsilon^2(\tau)}$ FOR DIFFERENT β
⊙ ⟵ Rectangular Pulse
△ ⟵ 3-bit Barker code
□ ⟵ 11-bit Barker code
1
2
3
4
5
6
7
8
9
10
20
30
40
50
100
200
300
400
500
0
1
2
3
4
5
$\beta = \tau_0/T$

for the rectangular pulse has no real peak. But while the mean-square error about a side peak will be small for the 3-bit code, we are almost certain to make a catastrophic error. This, of course, limits the meaningfulness of the results in this section.

V. CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

The results of the previous chapter showed to a good degree of accuracy how y(0) changes as a function of the dispersion ratio β, what the entire correlator output $y(\tau)$ looks like for several different values of β, and to a lesser degree of accuracy", how the mean-square estimation error changes as a function of β. I would like to stress the "lesser degree of accuracy in the curves relating $\overline{\varepsilon^2(\tau)}$ to β. One obvious problem, of course, is the limited number of points on which the curves are based. The accuracy of the points themselves is also in doubt in some instances. In particular I question the validity of the curve for the 11-bit code. I am at a loss to account for the behavior of this curve, especially when compared to the other two. Also the computations for the 11-bit code are most susceptible to machine round-off error because of the larger number of data points required. I suggest that the calculations that go from $y(\tau)$ to $\overline{\varepsilon^2(\tau)}$ be re-done using more data points, and then checked by hand.

By now the reader should be aware of some of the undesirable features of the method used to calculate $y(\tau)$. The problem is not the limit of 1000 data points in MAP (this can be overcome); the problem is that so many data points are needed in the first place. This problem is inherent in the fact that we are calculating $y(\tau)$ from equation 68. The sine and cosine terms in this equation oscillate more and more rapidly as v gets larger (especially for small τ_o) so that an enormous number of points must be used to evaluate the integrals. And the magnitude of this problem grows exponentially as the range of the auto-correlation function R(v) is made longer. For example, to calculate $y(\tau)$ for a doubly-folded 11-bit Barker code (121 elementary pulses), with $\beta = 1$, we would need at least 20,000 data points to evaluate the integrals. When we consider that each integral will have to be evaluated at least 500 times (because of the larger range of τ), we can begin to see what this means in terms of an expenditure in computer time.

Appendix C contains a derivation for an alternative expression for $y(\tau)$ consisting of sums of Fresnel integrals. These Fresnel integrals have been well tabulated (e.g. Ref. 1), thus suggesting a way of calculating $y(\tau)$ without performing any integrations. The problem, of course, is the necessity of feeding an entire table of Fresnel integrals into the computer and allocating storage space for it. Once this has been done, though, the stored table could be used to solve any number of cases. For simple cases (including the 11-bit code) this really is not worth it and it is more efficient to calculate $y(\tau)$ from equation 68 as I have done. However, for more complicated cases (doubly and triply-folded codes) this alternative approach is much more reasonable, and in fact is almost necessary. I would suggest, then, that a program be written to calculate $y(\tau)$ in this way.

Another point that should be raised is the question of how these results relate to the impending Sunblazer experiment. In particular, what kind of numbers can we expect for the dispersion time τ_o for different points in the Sunblazer trajectory. In Appendix A, I have used the Allen-Baumbach model for the extended corona to calculate a curve for τ_o as a function of path offset for the proposed Sunblazer orbit. With a signal baud length of 20 microseconds, the curve tells us that β will not exceed unity until the path offset becomes less than two solar radii. The reader should realize, however, that the significance of this curve is dubious. To begin with, the Allen-Baumbach model only gives a very rough estimate (accurate to a factor of three or four) of the average value of the coronal electron density. Solar prominences and bursts can cause the electron density to change by orders of magnitude over short lengths of time, and this can occur frequently during periods of high solar activity.

It is this fact that we cannot predict how much dispersion to expect ahead of time that puts a limit on the performance of any communication scheme that we might design. That is, if we knew exactly what τ_o would be at every point in the satellite trajectory, then we could re-design our receiver so as to again achieve optimum performance. For example, we could disperse the internally generated m(t) by an equivalent amount so that an auto-correlation function would again result (this will give optimum detection performance, but the estimation performance would still be somewhat degraded, though less so than before). Or we could put a pre-filter in front of the receiver that would "undisperse" the received waveform. If this filter was reversible we could recalculate the decision rules so as to again have optimum receiver performance. The point is that such a scheme would not work because we would not know how much dispersion to compensate for.

The fact that we are stuck with a dispersive channel for which we cannot compensate suggests further work directed towards tying dispersion into the communication problem. For example, I have mentioned that τ_o is a measure of an effective bandwidth of the medium, and if we try to transmit a signal with bandwidth larger than this it will be dispersed such that the resulting bandwidth will be smaller than the medium's bandwidth. This in turn places a limit on the estimation error than we can hope to achieve. I have not, however, tried to analyze this in any exact, quantitative terms, but this should be done in the future. Another problem that should be examined is to see how coding is related to dispersion. For example, is there a class of codes, or could a class of codes be devised, that behaves particularly well in a dispersive channel?

The result of folding a code on itself should also be examined in the light of dispersion. As a step in this direction, I plan in the near future to calculate $y(0)$, $y(\tau)$, and $\overline{\varepsilon^2(\tau)}$ for at least the doubly-folded 5-bit Barker code.

VI. APPENDICES

A. Calculation of τ_o for the Sunblazer Trajectory.

We begin with the dispersion equation for the medium (equation 55):

$$k(\omega) = \frac{1}{c}\,(\omega^2-\omega_p^2)^{1/2}$$

then $$k''(\omega) = \frac{1}{c}\,(\omega^2-\omega_p^2)^{-1/2} - \frac{\omega^2}{c}\,(\omega^2-\omega_p^2)^{-3/2} \tag{69}$$

$$(\omega^2-\omega_p^2)^{-1/2} \approx \frac{1}{\omega}\,(1 + \frac{\omega_p^{\,2}}{2\omega^2})$$

$$\omega^2(\omega^2-\omega_p^{\,2})^{-3/2} \approx \frac{1}{\omega}\,(1 + \frac{3\omega_p^{\,2}}{2\omega^2})$$

then,

$$k''(\omega_o) \approx \frac{1}{c}\,[\frac{1}{\omega_o} + \frac{\omega_p^{\,2}}{2\omega_o^{\,3}} - \frac{1}{\omega_o} - \frac{3\omega_p^{\,2}}{2\omega_o^{\,3}}]$$

$$= -\frac{\omega_p^{\,2}}{c\omega_o^{\,3}} \tag{70}$$

$$\omega_p^{\,2} = 3.16\cdot10^3 n, \text{ n in electrons/m}^2$$

Take f_o = 75 Mcps, $\omega_o = 4.72\cdot10^8$ rad/sec, then

$$|k''(\omega_o)| = 1.0\cdot10^{-32} n \tag{71}$$

The dispersion time τ_o has been defined as:

$$\tau_o = \sqrt{\pi\phi''(\omega_o)} = \sqrt{\pi\,|k''(\omega_o)|\,z} \tag{72}$$

where z is the path length. This expression must be integrated over the propagation path (see figure 1):

$$\tau_o = \left\{ \int \pi \cdot 10^{-32} \, n(r) \, ds \right\}^{1/2} \tag{73}$$

$$r^2 = s^2 + \rho^2, \; ds = \frac{rdr}{\sqrt{r^2 - \rho^2}}$$

For a particular point in the trajectory, let R_e be the distance from the sun to the earth, and R_s from the sun to the satellite. Then,

$$\int n(r)ds = \int_{\rho}^{R_s} \frac{n(r) \cdot rdr}{\sqrt{r^2 - \rho^2}} + \int_{\rho}^{R_s} \frac{n(r) \cdot rdr}{\sqrt{r^2 - \rho^2}} \tag{74}$$

If we want to express r in solar radii, then we multiply each integral by R_o. We use the Allen-Baumbach model for the extended corona:

$$n(r) = \frac{10^{14}}{r^6} + \frac{10^{12}}{r^2} \text{ electrons/m}^2,$$

$$r \text{ in solar radii, } r \geq 1.5 \, R_o$$

If we substitute this into equation 74 and evaluate the integrals we find that:

$$R_o \int_{\rho}^{x} \frac{n(r)rdr}{\sqrt{r^2 - \rho^2}} = 10^{14} \cdot R_o \left[\frac{\sqrt{x^2-\rho^2}}{4x^2\rho^2} + \frac{3\sqrt{x^2-\rho^2}}{8x^2\rho^4} + \frac{3}{8\rho^5} \tan^{-1}\left(\frac{x}{\rho}\right) \right] +$$

$$10^{12} \, R_o \cdot \frac{1}{\rho} \cos^{-1}\left(\frac{\rho}{x}\right) \tag{75}$$

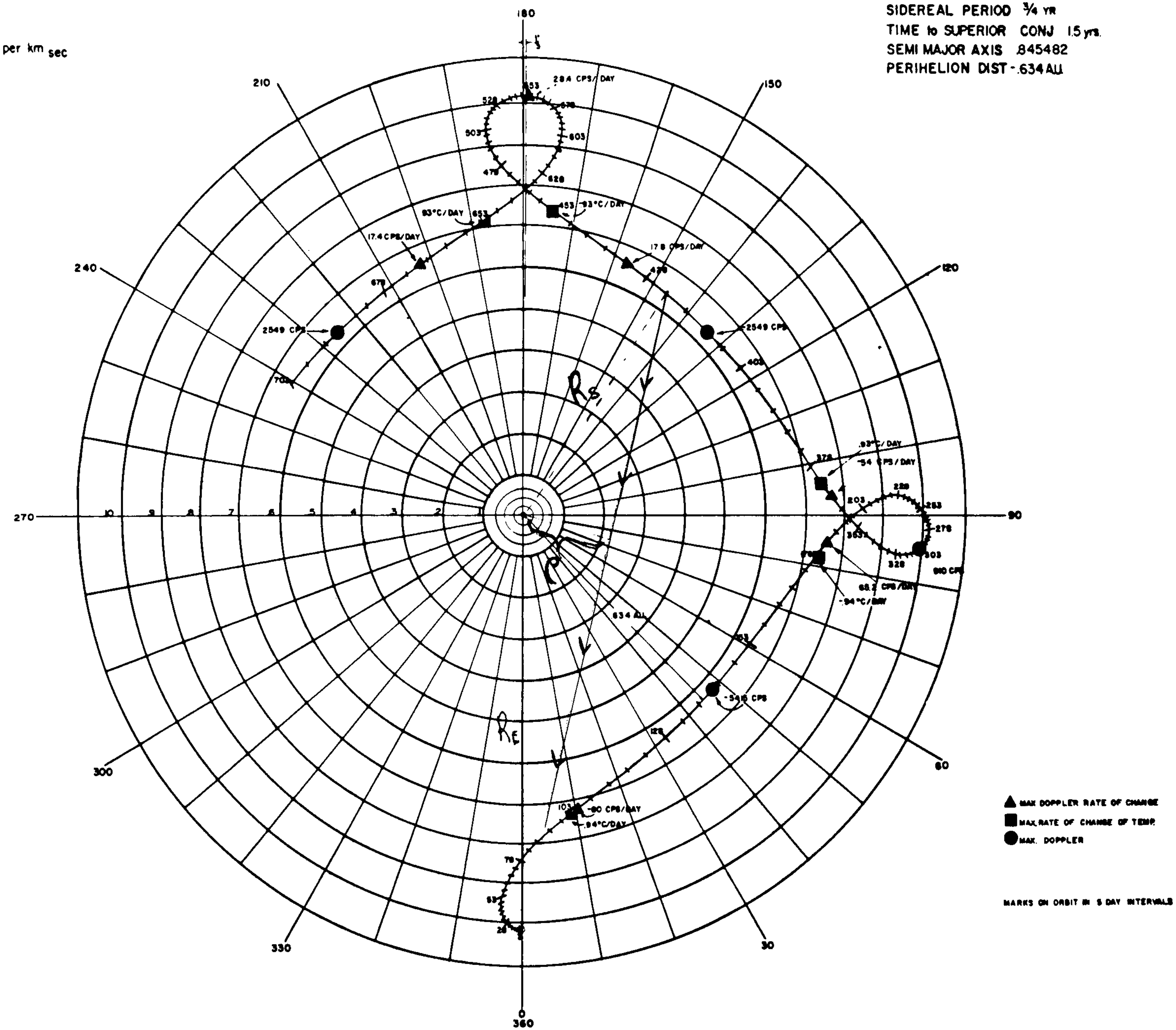
Here x is either R_e or R_s. To good approximation the first three terms on the right can be ignored for $\rho > 6R_o$. We have then,

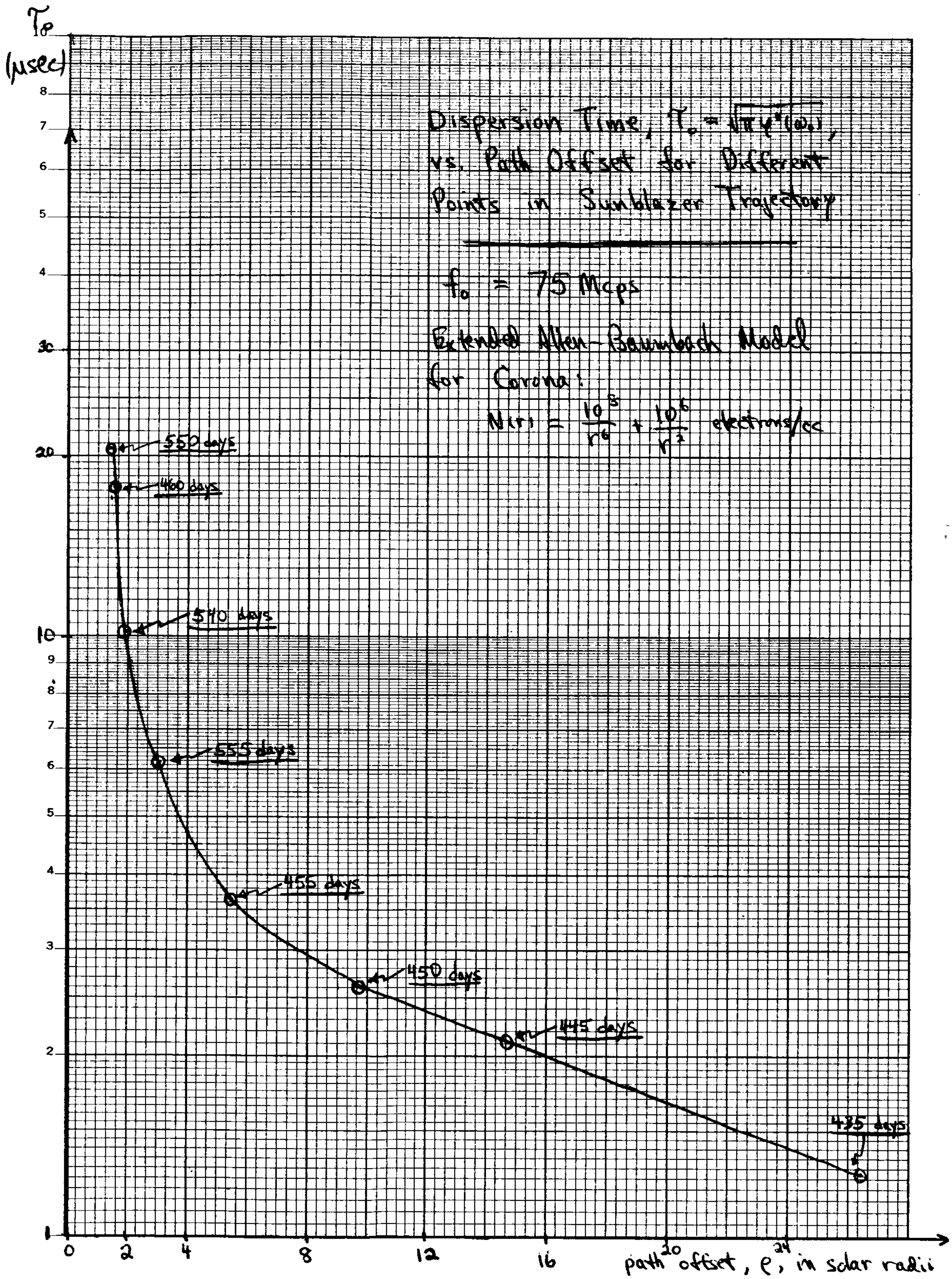
$$\tau_o = \left\{ \pi \cdot 10^{-32} R_o \left[\int_{\rho}^{R_s} + \int_{\rho}^{R_e} \right] \right\}^{1/2} \tag{76}$$

RELATIVE MOTION OF SOLAR PROBE

LAUNCH JULY 1
ECCEN. .231675
PER. SEN. 19.79 days per km sec

SIDEREAL PERIOD 3/4 YR
TIME to SUPERIOR CONJ 1.5 yrs.
SEMI MAJOR AXIS .845482
PERIHELION DIST - .634 AU





τ_0 (μsec)
Dispersion Time, $\tau_0 = \sqrt{\pi \psi''(\omega_0)}$, vs. Path Offset for Different Points in Sunblazer Trajectory
f_0 = 75 Mcps
Extended Allen-Baumbach Model for Corona:
$N(r) = \frac{10^8}{r^6} + \frac{10^6}{r^2}$ electrons/cc
550 days
460 days
540 days
555 days
455 days
450 days
445 days
435 days
1
2
3
4
5
6
7
8
9
10
20
30
0
2
4
8
12
16
20
24
path offset, ρ, in solar radii

Numerical values for τ_o were calculated using the data for the proposed Sunblazer orbit that were computed by Charles Peterson. A plot of τ_o as a function of ρ, together with Peterson's plot of the Sunblazer orbit appear on the preceding pages.

B. Attenuation Due to Collisions.

The coefficient of attenuation due to collisions follows from a second-order model for the medium. The equation of motion for the electrons will now have a damping term added on to it:

$$\underline{E}e = m\frac{d\underline{v}}{dt} + \alpha\underline{v} \tag{77}$$

Here $\alpha = m\nu$ where ν is the electron-ion collision frequency. Since $\underline{J} = ne\underline{v}$,

$$\frac{d\underline{J}}{dt} + \frac{\alpha}{m}\underline{J} = \frac{ne^2}{m}\underline{E} \tag{78}$$

Expressing $\underline{E}$ and $\underline{J}$ as complex exponentials for a plane wave, and writing the wave equation:

$$\underline{E} = \underline{i}_x E_o e^{j(\omega t - kz)} \tag{79a}$$

$$\underline{J} = \underline{i}_x J_o e^{j(\omega t - kz)} \tag{79b}$$

$$\frac{\partial^2 E}{\partial z^2} = \mu_o \frac{\partial J}{\partial t} + \mu_o \varepsilon_o \frac{\partial^2 E}{\partial t^2} \tag{80}$$

Finally, we make the approximation of very small attenuation, namely that $\nu/\omega \lll 1$.

Now substituting equations 79 into equation 78 gives:

$$J_o = \frac{ne^2}{\alpha + j\omega} E_o \tag{81}$$

and putting (79) into (80) gives:

$$-k^2 E_o = -\mu_o \varepsilon_o \omega^2 E_o + j\omega\mu_o J_o \tag{82}$$

Combining (81) and (82), we have:

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{\mu_o ne^2 m\omega^2}{\alpha^2 + \omega^2 m^2} - \frac{j\mu_o ne^2 \alpha\omega}{\alpha^2 + \omega^2 m^2}} \tag{83}$$

We can then manipulate this expression and make an approximation for the square root. The coefficient of attenuation, κ, is just the negative of the imaginary part of k. Omitting the algebra, we have:

$$\kappa = \frac{1}{2}\mu_o ne^2 c\alpha[(\alpha^2 + m^2\omega^2 - \mu_o ne^2 mc^2)(\alpha^2 + m^2\omega^2)]^{-1/2}$$

$$= \frac{\alpha\omega_p^{\ 2}}{2mc\omega}\left(\frac{\alpha^4}{\omega^2 m^4} + 2\frac{\alpha^2}{m^2} - \frac{\omega_p^{\ 2}\alpha^2}{\omega^2 m^2} + \omega^2 - \omega_p^{\ 2}\right)^{-1/2} \tag{84}$$

Taking $\nu \simeq 10^{-7}$ cps, $\omega \simeq 10^8$cps, we see that

$$\frac{\alpha^2}{m^2} \simeq 0; \quad \frac{\alpha^4}{\omega^2 m^4} = \frac{\nu^4}{\omega^2} \simeq 0; \text{ and } \frac{\omega_p^{\ 2}\alpha^2}{\omega^2 m^2} = \frac{\omega_p^{\ 2}}{\omega^2}\nu^2 \simeq 0.$$

We have, then,

$$\kappa = \frac{\alpha\omega_p^{\ 2}}{2mc\omega} \cdot \frac{1}{\sqrt{\omega^2 - \omega_p^{\ 2}}}$$

$$= \frac{\nu}{2c} \cdot \frac{(\omega_p/\omega)^2}{\sqrt{1-(\omega_p/\omega)^2}} \tag{85}$$

An alternative derivation is outlined in Ref. 3.

C. Another Method of Calculating $y(\tau)$ for Binary Signals.

Binary Signals

If m(t) is a binary waveform, the auto-correlation function can be expressed as:

$$R_{mm}(\tau) = \sum_{k=-N}^{N-1} (a_k\tau + b_k)\Big|_{kT}^{(k+1)T} \tag{86}$$

Here N is the number of binary bits, and $(a_k\tau+b_k)\Big|_{kT}^{(k+1)T}$

is a line segment defined over the interval $(k+1)T \leq \tau \leq kT$. The three-bit Barker code is shown as an example in figure 12a.

We want to evaluate an integral of the form:

$$\int_{-\infty}^{\infty} R_{mm}(\tau-\tau_o u)e^{-j\frac{\pi}{2}u^2}du$$

Using the same a_k's and b_k's we can write $R_{mm}(\tau-\tau_o u)$ as a function of u (figure 12b):

$$R_{mm}(\tau-\tau_o u) = \sum_{k=-N}^{N-1} (a_k\tau_o u+b_k-a_k\tau)\Bigg|_{u=\frac{\tau+kT}{\tau_o}}^{\frac{\tau+(k+1)T}{\tau_o}} \tag{87}$$

Then,

$$\int_{-\infty}^{\infty} R_{mm}(\tau-\tau_o u)e^{-j\frac{\pi}{2}u^2}du =$$

$$\sum_{k=-N}^{N-1} a_k\tau_o \int_{l_k}^{l_{k+1}} ue^{-j\frac{\pi}{2}u^2}du + \sum_{k=-N}^{N-1}(b_k-a_k\tau)\int_{l_k}^{l_{k+1}} e^{-j\frac{\pi}{2}u^2}du \tag{88}$$

where

$$l_k = \frac{\tau+kT}{\tau_o}$$

$$\int_{l_k}^{l_{k+1}} ue^{-j\frac{\pi}{2}u^2}du = \frac{j}{\pi}\left[\cos\frac{\pi}{2}l_{k+1}^2 - \cos\frac{\pi}{2}l_k^2\right] + \frac{1}{\pi}\left[\sin\frac{\pi}{2}l_{k+1}^2-\sin\frac{\pi}{2}l_k^2\right]$$

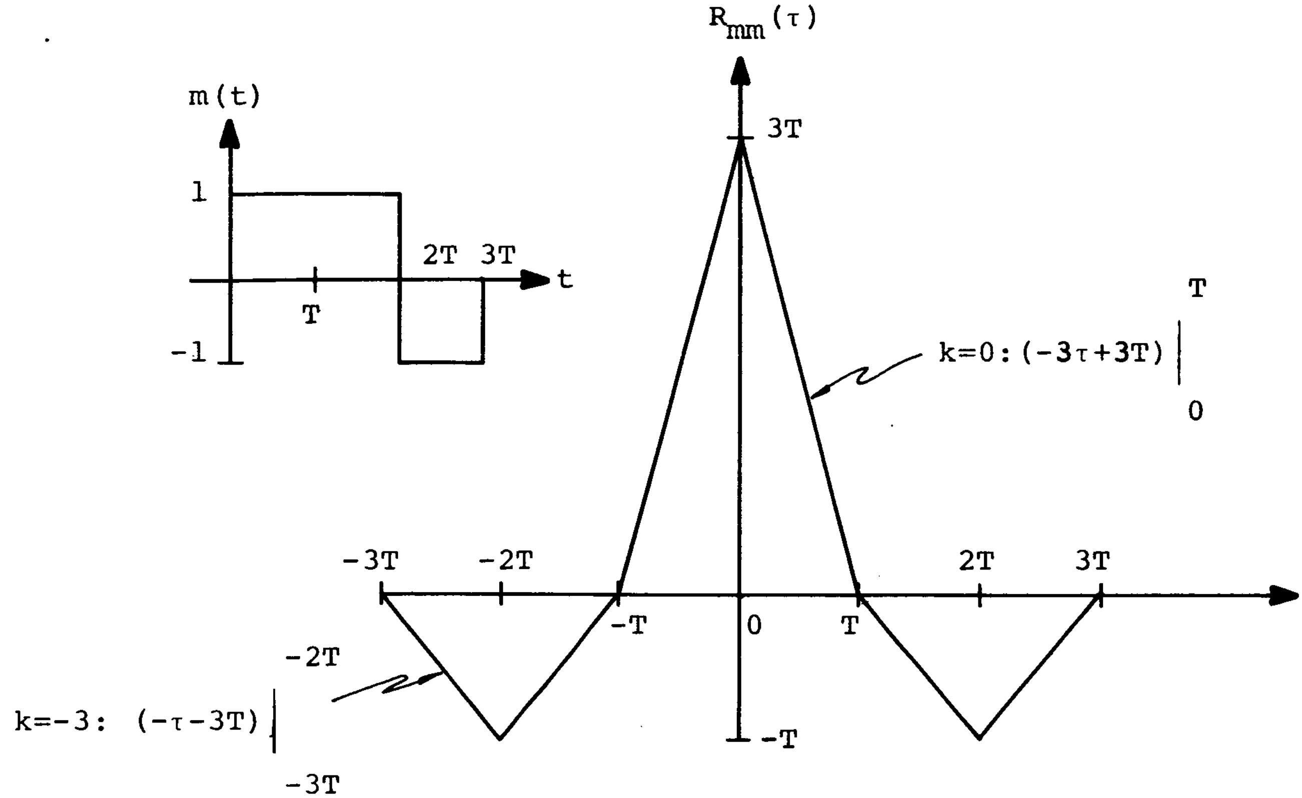


Figure 12a

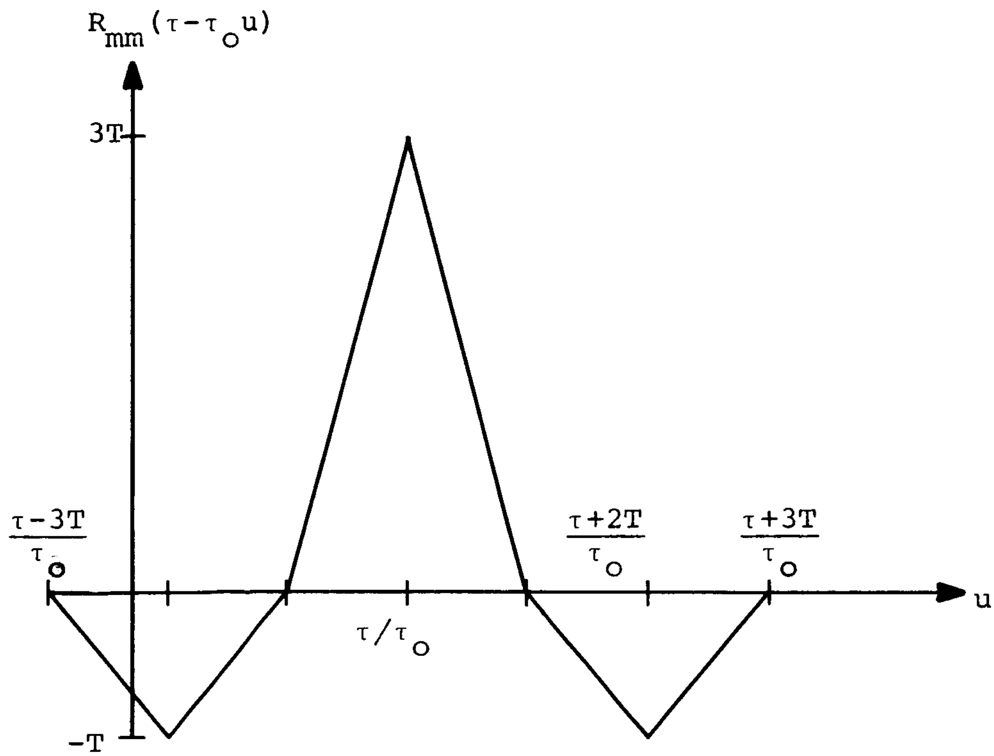


Figure 12b

and $$\int_{l_k}^{l_{k+1}} e^{-j\frac{\pi}{2}u^2} du = F^*(l_{k+1}) - F^*(l_k)$$

$$= [C(l_{k+1})-C(l_k)]-[S(l_{k+1})-S(l_k)]$$

Here $C(x)$ and $S(x)$ are the Fresnel cosine and sine integrals:

$$C(x) = \int_o^x \cos \tfrac{\pi}{2}\xi^2 d\xi; \quad S(x) = \int_o^x \sin\tfrac{\pi}{2}\xi^2 d\xi$$

Rewriting equation 67:

$$y(\tau) = \frac{1}{\sqrt{2}}\left\{\left[\mathrm{Re} \int_{-\infty}^{\infty} R_{mm}(\tau-\tau_o u) e^{-j\frac{\pi}{2}u^2} du\right]^2 + \left[\mathrm{Im} \int_{\infty}^{\infty} R_{mm}(\tau-\tau_o u) e^{-j\frac{\pi}{2}u^2} du\right]^2\right\}^{\frac{1}{2}}$$

We see, then, that

$$y(\tau) = \frac{1}{\sqrt{2}}\left\{\left[\sum_{k=-N}^{N-1} P_k(\tau)\right]^2 + \left[\sum_{k=-N}^{N-1} Q_k(\tau)\right]^2\right\}^{1/2} \tag{89}$$

where

$$P_k = \frac{a_k\tau_o}{\pi} [\sin\tfrac{\pi}{2}l_{k+1}^2-\sin\tfrac{\pi}{2}l_k^2] + (b_k-a_k\tau)[C(l_{k+1})-C(l_k)] \tag{90a}$$

and

$$Q_k = \frac{a_k\tau_o}{\pi} [\cos\tfrac{\pi}{2}l_{k+1}^2-\cos\tfrac{\pi}{2}l_k^2]-(b_k-a_k\tau)[S(l_{k+1})-S(l_k)] \tag{90b}$$

Since $C(x)$ and $S(x)$are well tabulated, equations 89 and 90 provide a way of calculating $y(\tau)$ without performing any integrations. The accuracy of the calculation will depend on the size of the Fresnel integral table that is fed into the computer, but once the table is fed in, the same accuracy will be attained for a binary signal of any length. This method is particularly advantageous, then, for analyzing longer and more complex signals.

D. Computer Programs

The next page contains the MAD programs to calculate $y(0)$ and $y(\tau)$. The page following contains a sample command-response exchange in MAP for the calculation of $\overline{\varepsilon^2(\tau)}$.

```
******************************************************************************************
M5288        3256     CRCSO          MAD FOR   M5288        3256     04/24       2538.1
             DIMENSION R(1000),Y(100)
             INTEGER MINV,MAXV,J,I,MINT,MAXT,MINW,MAXW
             EXECUTE IN.($R(V)*$,R,MINV,MAXV,DELV)
             TO=VALUE.($TO*$)
             TOS=TO.P.2
             INTEGA=0
             INTEGB=0
             THROUGH LOOPB, FOR I=MINV,1,I.G.MAXV
             V=DELV*I
             ARG=1.5708*(V).P.2/TOS
             INTEGA=INTEGA+R(I-MINV)*COS.(ARG)
LOOPB        INTEGB=INTEGB+R(I-MINV)*SIN.(ARG)
             Y=.70711*SQRT.(DELV.P.2*(INTEGA.P.2+INTEGB.P.2)/TOS)
             PRINT RESULTS Y
             EXECUTE CONST.($Y*$,Y)
             EXECUTE CHNCOM.
             END OF PROGRAM
FILE
```

```
******************************************************************************************
M5288        3256     CRSCOR         MAD FOR   M5288        3256     04/24       2538.1
             DIMENSION R(1000),Y(1000),C(1000),S(1000)
             INTEGER MINV,MAXV,J,I,MINT,MAXT,MINW,MAXW
             EXECUTE IN.($R(V)*$,R,MINV,MAXV,DELV)
             EXECUTE RANGE.($T*$,MINT,MAXT,DELT)
             TO=VALUE.($TO*$)
             TOS=TO.P.2
             THROUGH LOOPA,FOR J=MINT,1,J.G.MAXT
             T=J*DELT
             INTEGA=0
             INTEGB=0
             THROUGH LOOPB, FOR I=MINV,1,I.G.MAXV
             V=DELV*I
             ARG=1.5708*(V+T).P.2/TOS
             INTEGA=INTEGA+R(I-MINV)*COS.(ARG)
LOOPB        INTEGB=INTEGB+R(I-MINV)*SIN.(ARG)
LOOPA        Y(J-MINT)=.70711*SQRT.(DELV.P.2*(INTEGA.P.2+INTEGB.P.2)/TOS)
             EXECUTE CUT.($Y(T)*$,Y,MINT,MAXT,DELT)
             EXECUTE CHNCOM.
             END OF PROGRAM
FILE
```

```
COMMAND PLEASE
delete to

COMMAND PLEASE
delete minmax t

COMMAND PLEASE
execute crscor
PLEASE PRINT ON NEXT LINE MIN, MAX, AND DEL FOR THE VARIABLE    T .
-.0006 .0006 .000008
MIN = -   75      MAX =     75
DEC. VALUE OF CONSTANT      TO PLEASE.    .00002

COMMAND PLEASE
basis p(v) p(v)
PLEASE GIVE THE NAME OF THE CHANGE-OF-BASIS FUNCTION OR THE NEW VALUE OF DEL  .000008

COMMAND PLEASE
plot y(t) p(v) 0.0 .0006

COMMAND PLEASE
basis p(v) p(v)
PLEASE GIVE THE NAME OF THE CHANGE-OF-BASIS FUNCTION OR THE NEW VALUE OF DEL  .000004

COMMAND PLEASE
transform y(t)
WHAT WOULD YOU LIKE TO CALL THE SINE TRANSFORM
WHAT WOULD YOU LIKE TO CALL THE COSINE TRANSFORM      c(w)

COMMAND PLEASE
integrate 1 c(w) intn
THE VALUE OF THE LOWER LIMIT PLEASE =     0.0
THE VALUE OF THE UPPER LIMIT PLEASE =     900000.

COMMAND PLEASE
print intn
  INTN =      .44020E-03

COMMAND PLEASE
(x(w)=c(w)*w**2)

COMMAND PLEASE
integrate 1 x(w) intw
THE VALUE OF THE LOWER LIMIT PLEASE =    0.0
THE VALUE OF THE UPPER LIMIT PLEASE =     900000.

COMMAND PLEASE
print intw
  INTW =      .14387E 07

COMMAND PLEASE
basis y(t) y(t)
PLEASE GIVE THE NAME OF THE CHANGE-OF-BASIS FUNCTION OR THE NEW VALUE OF DEL  .000004

COMMAND PLEASE
plot y(t) p(v) 0.0 .0006
```

Example of Calculation of $y(\tau)$, $\overline{\varepsilon^2}(\tau)$, using MAP

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